Integration on Groups and Non-abelian Fourier Analysis

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The primary purpose of these notes is to give a self-contained discussion on the construction of Haar measures and Haar integrals on locally compact Hausdorff groups. The most important examples one should keep in mind are finite dimensional Lie groups such as the special linear group $\mathrm{SL}_2(\mathbb{R})$. As an application of these basic tools, we develop the elegant Fourier analysis (the L^2-theory) on compact Hausdorff groups (the Peter-Weyl theorem), which generalises the classical theory of Fourier series on the unit circle (the abelian case). Our approach to Haar integration follows the main lines of [3, 4]. Our discussion on non-abelian Fourier analysis is largely inspired by [1, 5, 6].

Contents

1 The Riesz-Markov-Kakutani representation theorem

Formally speaking, a measure μ on a space X induces a linear functional

$$\lambda_\mu : f \mapsto \int_X f d\mu$$

on a suitable space $\mathcal{C}$ of functions over X. This functional is positive in the sense that

$$f \geqslant 0 \implies \lambda_\mu(f) \geqslant 0.$$

The Riesz-Markov-Kakutani representation theorem asserts that the converse is also true, namely, all positive linear functionals on $\mathcal{C}$ arise in this way. This gives us a powerful way of constructing measures on X from the duality viewpoint of linear functionals, which is also the approach of constructing Haar measures on locally compact Hausdorff groups that we will adopt in the present notes.

To set up the framework properly, throughout the rest of this section we assume that X is a *locally compact, Hausdorff* space and $\mathcal{B}(X)$ denotes the Borel σ-algebra over X (i.e. the σ-algebra generated by open subsets of X). We use $\mathcal{C}_c(X)$ to denote the space of complex-valued continuous functions on X with compact support.

Remark 1.1. Working with locally compact Hausdorff spaces allows us to construct sufficiently many continuous functions with compact support. These functions play essential roles in our study.

Definition 1.1. A measure $\mu : \mathcal{B}(X) \to [0, \infty]$ is called a *Radon measure* if it satisfies the following properties:

(i) *Outer regularity*: for any $A \in \mathcal{B}(X)$, one has

$$\mu(A) = \inf\{\mu(V) : V \text{ open}, V \supseteq A\}; \tag{1.1}$$

(ii) *Inner regularity*: for any open subset V, one has

$$\mu(V) = \sup\{\mu(K) : K \text{ compact}, K \subseteq V\};$$

(iii) *Finiteness on compact sets*:

$$\mu(K) < \infty \text{ for any compact subset } K.$$

Remark 1.2. In the same way, one can also define the notion of Radon measures on σ-algebras containing $\mathcal{B}(X)$.

Definition 1.2. A *positive linear functional* λ on $\mathcal{C}_c(X)$ is a (complex-valued) linear functional $\lambda : \mathcal{C}_c(X) \to \mathbb{C}$ such that

$$f \text{ real}, \ f \geqslant 0 \implies \lambda(f) \geqslant 0.$$

Let μ be a Radon measure on X. Then μ induces a linear functional on $\mathcal{C}_c(X)$ defined by integration:

$$\lambda_\mu(f) \triangleq \int_X f d\mu, \quad f \in \mathcal{C}_c(X). \tag{1.2}$$

This functional is clearly positive. The Riesz-Markov-Kakutani representation theorem (in what follows, we simply refer to it as the RMK representation theorem) asserts that every positive linear functional on $\mathcal{C}_c(X)$ is the integration against a Radon measure.

Theorem 1.1. *Let $\lambda : \mathcal{C}_c(X) \to \mathbb{C}$ be a positive linear functional. Then there exists a unique Radon measure μ on $\mathcal{B}(X)$, such that*

$$\lambda(f) = \int_X f d\mu, \quad f \in \mathcal{C}_c(X).$$

1.1 Some notation and a key property of Radon measures

We start by discussing a key property of Radon measures which motivates the strategy of proving the representation theorem.

Let us first introduce some notation which will be used throughout the rest. Let K be a compact subset of X. We use $K \prec f$ to mean that

$$f \in \mathcal{C}_c(X), \ 0 \leqslant f \leqslant 1 \text{ on } X \text{ and } f = 1 \text{ on } K. \tag{1.3}$$

Similarly, let V be an open subset of X. We use $f \prec V$ to mean that

$$f \in \mathcal{C}_c(X), \ 0 \leqslant f \leqslant 1 \text{ on } X \text{ and } f \text{ is supported in } V.$$

As a result, given $K \subseteq V$, the notation $K \prec f \prec V$ means

$$f \in \mathcal{C}_c(X), \ \mathbf{1}_K \leqslant f \leqslant \mathbf{1}_V.$$

The following lemma is a simple application of Urysohn's lemma (cf. Theorem B.2 in Appendix B). It is a useful tool for producing rich continuous functions on topological spaces.

Lemma 1.1. *Let $K \subseteq V$ where K is compact and V is open. Then there exists $f \in \mathcal{C}_c(X)$ such that $K \prec f \prec V$.*

Proof. We first claim that, there exists an open subset W such that $K \subseteq W \subseteq \overline{W} \subseteq V$ and $\overline{W}$ is compact. Indeed, for each $x \in K$ one finds a compact neighbourhood W_x of x such that $\overline{W_x} \subseteq V$. By the compactness of K, one covers K by finitely many such W_x's, whose union gives the desired open subset W.

Now since $\overline{W}$ is a compact Hausdorff space (hence normal), one can apply Urysohn's lemma on it. More precisely, note that $W^c \cap \overline{W}$ is a closed subset disjoint from K. According to Urysohn's lemma, one finds a continuous function $g : \overline{W} \to [0,1]$ such that $g = 1$ on K and $g = 0$ on $W^c \cap \overline{W}$. The function g trivially extends to a continuous function $f : X \to [0,1]$ by setting $f = 0$ on $\overline{W}^c$. The function f satisfies the desired properties. □

Remark 1.3. The technical complication in the above proof comes from the fact that a locally compact Hausdorff space may fail to be normal in general. Therefore, some care is needed when applying Urysohn's lemma.

The next property is the key observation for motivating the proof of Theorem 1.1.

Proposition 1.1. *Let μ be a Radon measure on X and let V be an open subset of X. Then one has*

$$\mu(V) = \sup \big\{ \lambda_\mu(f) : f \prec V \big\}, \tag{1.4}$$

where λ_μ is the linear functional defined by (1.2).

Proof. It is trivial that $\mu(V)$ is an upper bound of $\lambda_\mu(f)$ for any $f \prec V$. To see that it is indeed the supremum, we only consider the case when $\mu(V) < \infty$ and leave the similar discussion of the other case to the reader. According to inner regularity, given $\varepsilon > 0$, there exists a compact subset $K \subseteq V$ such that

$$\mu(K) > \mu(V) - \varepsilon.$$

By using Lemma 1.1, one can choose an $f \in \mathcal{C}_c(X)$ such that $K \prec f \prec V$. It follows that

$$\lambda_\mu(f) \geqslant \mu(K) > \mu(V) - \varepsilon.$$

Therefore, $\mu(V)$ is the supremum of $\{\lambda_\mu(f) : f \prec V\}$. □

The above result gives the uniqueness part of Theorem 1.1 trivially. Indeed, if μ and ν are two Radon measures satisfying the theorem, the property (1.4) implies that $\mu = \nu$ on open subsets. According to the outer regularity property, one concludes that $\mu = \nu$ on $\mathcal{B}(X)$.

1.2 Proof of the representation theorem

The property (1.4) is also the key for motivating the proof of existence. The main strategy can be summarised as follows. Let $\lambda : \mathcal{C}_c(X) \to \mathbb{C}$ be a given positive linear functional.

Step one. Use the formula (1.4) and the outer regularity property to construct an outer measure μ on all subsets of X.
Step two. Identify a suitable σ-algebra $\mathcal{M}$ that contains $\mathcal{B}(X)$, such that the restriction of μ on $\mathcal{M}$ is a Radon measure.
Step three. Show that the linear functional induced by integration against μ coincides with λ.

In what follows, we develop these three steps carefully.

1.2.1 Step one: constructing the outer measure

Inspired by (1.4) and the outer regularity property (1.1), we define a set function μ on all subsets of X by

$$\mu(A) \triangleq \inf\{\mu(V) : V \text{ open}, V \supseteq A\}, \quad A \subseteq X,$$

where for any open subset V we define

$$\mu(V) \triangleq \sup\left\{\lambda(f) : f \prec V\right\}.$$

Lemma 1.2. *The set function μ is an outer measure over X.*

The proof of Lemma 1.2 relies on Dini's theorem and a continuity property of positive linear functions which we now discuss.

Theorem 1.2 (Dini's theorem)**.** *Let $f \in \mathcal{C}_c(X)$ be a given non-negative function. Let Φ be a family of non-negative functions in $\mathcal{C}_c(X)$ which satisfies the following properties:*

(i) for any $\varphi \in \Phi$, one has $\varphi \leqslant f$;
(ii) for any $\varphi, \psi \in \Phi$, one has $\varphi \vee \psi \triangleq \max\{\varphi, \psi\} \in \Phi$;
(iii) Φ approximates f pointwisely in the sense that

$$\sup_{\varphi \in \Phi} \varphi(x) = f(x) \quad \forall x \in X.$$

Then for any $\varepsilon > 0$, there exists $\varphi \in \Phi$, such that

$$\|f - \varphi\|_\infty < \varepsilon.$$

Proof. Let K be the support of f. According to Property (iii), for each $x \in K$, there exists $\varphi_x \in \Phi$ such that

$$f(x) - \varphi_x(x) < \varepsilon.$$

By continuity, one finds an open neighbourhood V_x of x, such that

$$f(y) - \varphi_x(y) < \varepsilon \quad \forall y \in V_x.$$

Since K is compact, it is covered by finitely many such neighbourhoods, say $V_{x_1}, \cdots, V_{x_n}$. Define

$$\varphi \triangleq \max\{\varphi_{x_1}, \cdots, \varphi_{x_n}\}.$$

Note that $\varphi \in \Phi$ by Property (ii). For any $y \in K$, one has $y \in V_{x_i}$ for some i, and thus

$$0 \leqslant f(y) - \varphi(y) \leqslant f(y) - \varphi_{x_i}(y) < \varepsilon.$$

□

Remark 1.4. A useful form of Dini's theorem is the case when Φ is given by an increasing sequence of non-negative $\mathcal{C}_c(X)$-functions that approaches f pointwisely. The theorem says that one automatically has uniform convergence for the family.

Proposition 1.2. *Let $\lambda : \mathcal{C}_c(X) \to \mathbb{C}$ be a positive linear functional. For any compact subset K, let $\mathcal{C}_K(X)$ be the subspace of $\mathcal{C}_c(X)$ containing those functions supported in K. Then $\lambda|_{\mathcal{C}_K(X)}$ is a continuous linear functional.*

Proof. By considering the real and imaginary parts separately, one may assume without loss of generality that λ and the involved functions are real-valued. Choose a function $g \in \mathcal{C}_c(X)$ such that $K \prec g$, whose existence is guaranteed by Lemma 1.1. For any real-valued $f \in \mathcal{C}_K(X)$, one has

$$-\|f\|_\infty g \leqslant f \leqslant \|f\|_\infty g.$$

It follows from the positivity of λ that

$$|\lambda(f)| \leqslant \|f\|_\infty \cdot \lambda(g).$$

Therefore, λ is bounded on $\mathcal{C}_K(X)$. □

Now we are able to give the proof of Lemma 1.2.

Proof of Lemma 1.2. The fact that $\mu(\emptyset) = 0$ and

$$A \subseteq B \implies \mu(A) \leqslant \mu(B)$$

are both obvious. The main challenge is to verify countable sub-additivity. We first establish finite sub-additivity on open subsets, and then extend this to the general case by a standard $\frac{\varepsilon}{2^n}$-series argument.

Let V_1, V_2 be two open subsets of X and let $f \prec V_1 \cup V_2$. One needs to show that

$$\lambda(f) \leqslant \mu(V_1) + \mu(V_2). \tag{1.5}$$

The idea is to use Dini's theorem for f. To this end, let

$$\Phi \triangleq \{\varphi_1 \vee \varphi_2 : \varphi_1 \prec V_1, \varphi_2 \prec V_2\},$$

and define

$$\Phi_f \triangleq \{f \wedge \varphi \triangleq \min\{f, \varphi\} : \varphi \in \Phi\}.$$

We first check that the family Φ_f satisfies the conditions of Dini's theorem. Property (i) is trivial. Property (ii) follows from the identity

$$\big(f \wedge (\varphi_1 \vee \varphi_2)\big) \vee \big(f \wedge (\psi_1 \vee \psi_2)\big) = f \wedge \big((\varphi_1 \vee \psi_1) \vee (\varphi_2 \vee \psi_2)\big),$$

where $\varphi_i, \psi_i \prec V_i$ $(i = 1, 2)$. As for Property (iii), fix $x \in K \triangleq \mathrm{supp} f$, and say $x \in V_1$. Choose $\varphi_1 \prec V_1$ such that $\varphi_1(x) = 1$. Then one has

$$\varphi \triangleq f \wedge (\varphi_1 \vee 0) \in \Phi_h, \ \varphi(x) = f(x).$$

Therefore, $f = \sup_{\varphi \in \Phi_h} \varphi$. According to Dini's theorem and the continuity of λ on $\mathcal{C}_K(X)$, one has

$$\begin{aligned}
\lambda(f) = \lambda\big(\sup_{\varphi \in \Phi_f} \varphi\big) = \sup_{\varphi \in \Phi_f} \lambda(\varphi) &= \sup_{\varphi_i \prec V_i \ (i=1,2)} \lambda(f \wedge (\varphi_1 \vee \varphi_2)) \\
&\leqslant \sup_{\varphi_i \prec V_i \ (i=1,2)} \lambda(f \wedge \varphi_1 + f \vee \varphi_2) \\
&\leqslant \sup_{\varphi_1 \prec V_1} \lambda(f \wedge \varphi_1) + \sup_{\varphi_2 \prec V_2} \lambda(f \wedge \varphi_2) \\
&\leqslant \mu(V_1) + \mu(V_2).
\end{aligned}$$

The claim (1.5) then follows.

Now we prove countable additivity. Let $\{A_n : n \geqslant 1\}$ be a sequence of subsets of X, and suppose that $\mu(A_n) < \infty$ for all n (the other case is trivial). By the definition of μ, for each ε and n there exists an open subset $V_n \supseteq A_n$ such that

$$\mu(V_n) < \mu(A_n) + \frac{\varepsilon}{2^n}.$$

Define the open subset $V \triangleq \cup_{n=1}^{\infty} V_n$. For any $f \prec V$, since f is compactly supported, there exists N such that

$$f \prec V_1 \cup \cdots \cup V_N.$$

It follows that

$$\begin{aligned}\lambda(f) &\leqslant \mu(V_1 \cup \cdots \cup V_N) \leqslant \mu(V_1) + \cdots + \mu(V_N) \\ &\leqslant \sum_{n=1}^{\infty} \big(\mu(A_n) + \frac{\varepsilon}{2^n}\big) = \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon.\end{aligned}$$

Since f is arbitrary, one concludes that

$$\mu\big(\cup_{n=1}^{\infty} A_n\big) \leqslant \mu(V) \leqslant \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon.$$

This gives the countable additivity property as ε is arbitrary.

□

1.2.2 Step two: restricting μ to a Radon measure

The next step is to restrict μ to a σ-algebra containing $\mathcal{B}(X)$ and show that the restriction is indeed a Radon measure. We could proceed along the abstract measure-theoretic approach of Carathéodory's extension theorem. However, this approach does not seem to be the simplest in the current context, as it is too general and does not reflect any topological considerations that are needed here. Instead, we try to write down the σ-algebra explicit in the way under which the Radon property becomes transparent.

Introduction of a ring

We first define the following class of subsets:

$$\mathcal{A} \triangleq \big\{A \subseteq X : \mu(A) < \infty \text{ and } \mu(A) = \sup\{\mu(K) : K \text{ compact}, K \subseteq A\}\big\}.$$

Lemma 1.3. *The set class $\mathcal{A}$ satisfies the following properties.*

(i) $\mathcal{A}$ contains the class of compact sets.
(ii) If V is open and $\mu(V) < \infty$, then $V \in \mathcal{A}$.
(iii) The restriction of μ on $\mathcal{A}$ is countably additive. More precisely, if $\{A_n : n \geqslant 1\}$ is a disjoint sequence of subsets in $\mathcal{A}$ and $A \triangleq \cup_{n=1}^{\infty} A_n$, then

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

In addition, if $\mu(A) < \infty$, one also has $A \in \mathcal{A}$.
(iv) $\mathcal{A}$ is a ring, i.e. $\emptyset \in \mathcal{A}$, and

$$A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A},\ A \backslash B \in \mathcal{A}.$$

Proof. (i) Let K be a compact subset. One only needs to show that $\mu(K) < \infty$. For this purpose, let V be an open subset such that $K \subseteq V$ and $\overline{V}$ is compact. Pick $g \in \mathcal{C}_c(X)$ such that $\overline{V} \prec g$. For any $f \prec V$ one has $f \leqslant g$. It follows from the definition of $\mu(V)$ that

$$\mu(K) \leqslant \mu(V) \leqslant \lambda(g) < \infty.$$

(ii) Let V be an open subset such that $\mu(V) < \infty$. Given $\varepsilon > 0$, one can find $f \prec V$ such that

$$\mu(V) < \lambda(f) + \varepsilon.$$

Since the support of f is a compact subset of V, one can find an open subset W such that $\overline{W}$ is compact and

$$f \prec W,\ \overline{W} \subseteq V.$$

As a result, one has

$$\mu(V) < \lambda(f) + \varepsilon \leqslant \mu(W) + \varepsilon \leqslant \mu(\overline{W}) + \varepsilon.$$

(iii) We first prove finite additivity on compact subsets, and then extend the result to the general case by a standard $\frac{\varepsilon}{2^n}$-series argument.

Let K_1, K_2 be two disjoint compact subsets. Pick two disjoint open subsets V_1, V_2 such that $K_i \subseteq V_i$ $(i = 1, 2)$. By the definition of $\mu(K_1 \cup K_2)$, there exists an open set W such that $W \supseteq K_1 \cup K_2$ and

$$\mu(W) < \mu(K_1 \cup K_2) + \varepsilon.$$

For $i = 1, 2$, we pick $g_i \prec W \cap V_i$ such that

$$\mu(W \cap V_i) - \varepsilon < \lambda(g_i).$$

It is clear that $g_1 + g_2 \prec W$. Therefore,

$$\mu(K_1) + \mu(K_2) - 2\varepsilon \leqslant \lambda(g_1) + \lambda(g_2) = \lambda(g_1 + g_2) < \mu(K_1 \cup K_2) + \varepsilon.$$

Since ε is arbitrary, one concludes that

$$\mu(K_1 \cup K_2) \geqslant \mu(K_1) + \mu(K_2).$$

Now we treat the general case. Let $\{A_n : n \geqslant 1\}$ be a disjoint sequence in $\mathcal{A}$ and $A \triangleq \cup_{n=1}^{\infty} A_n$. For each $n \geqslant 1$, one finds a compact subset $K_n \subseteq A_n$ such that

$$\mu(A_n) < \mu(K_n) + \frac{\varepsilon}{2^n}.$$

Note that the sequence $\{K_n : n \geqslant 1\}$ is disjoint and one can apply the finite additivity property just shown. It follows that

$$\sum_{i=1}^{n} \mu(A_i) \leqslant \sum_{i=1}^{n} \mu(K_i) + \varepsilon = \mu(K_1 \cup \cdots \cup K_n) + \varepsilon \leqslant \mu(A) + \varepsilon.$$

By letting $n \to \infty$ and $\varepsilon \to 0$, one obtains that

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

If $\mu(A) < \infty$, given $\varepsilon > 0$ one can find an n such that

$$\sum_{i=n+1}^{\infty} \mu(A_n) < \frac{\varepsilon}{2}.$$

For such n, one has

$$0 \leqslant \mu(A) - \mu(K_1 \cup \cdots \cup K_n) = \sum_{i=1}^{n} (\mu(A_i) - \mu(K_i)) + \sum_{i=n+1}^{\infty} \mu(A_i)$$

$$\leqslant \sum_{i=1}^{n} \frac{\varepsilon}{2^i} + \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, one concludes that $A \in \mathcal{A}$.

(iv) We only check the property that

$$A_1, A_2 \in \mathcal{A} \implies A_1 \backslash A_2 \in \mathcal{A}.$$

It is obvious that $\mu(A_1 \backslash A_2) < \infty$. Given $\varepsilon > 0$, by the definition of μ and $\mathcal{A}$, one finds an open subset $V_i \supseteq A_i$ as well as a compact subset $K_i \subseteq A_i$, such that

$$\mu(V_i) - \frac{\varepsilon}{2} < \mu(A_i) < \mu(K_i) + \frac{\varepsilon}{2}, \quad i = 1, 2.$$

In particular,

$$\mu(V_i) < \mu(K_i) + \varepsilon.$$

Observe that $K \triangleq K_1 \backslash V_2$ is compact subset contained in $A_1 \backslash A_2$. In addition, one has

$$(A_1 \backslash A_2) \backslash K \subseteq (A_1 \backslash K_1) \cup (V_2 \backslash A_2).$$

Therefore,

$$\begin{aligned}
0 &\leqslant \mu(A_1 \backslash A_2) - \mu(K) \leqslant \mu(A_1 \backslash K_1) + \mu(V_2 \backslash A_2) \\
&\leqslant \mu(V_1 \backslash K_1) + \mu(V_2 \backslash K_2) \\
&= \mu(V_1) - \mu(K_1) + \mu(V_2) - \mu(K_2) \quad \text{(by additivity proven in (iii))} \\
&< 2\varepsilon.
\end{aligned}$$

We thus conclude that $A_1 \backslash A_2 \in \mathcal{A}$. □

Remark 1.5. By essentially the same argument as in Part (ii), one can show that

$$\mu(V) = \sup\{\mu(K) : K \text{ compact}, \ K \subseteq V\}$$

for every open subset V. Namely, μ satisfies the inner regularity property.

Construction of the σ-algebra

We define

$$\mathcal{M} \triangleq \big\{Y \subseteq X : Y \cap K \in \mathcal{A} \text{ for all compact subset } K\big\}.$$

Note that the construction of $\mathcal{M}$ already encodes the Radon property, as seen from the definition of $\mathcal{A}$ and Lemma 1.3 (see also Remark 1.5). In addition, one has the following relation between $\mathcal{M}$ and $\mathcal{A}$.

Lemma 1.4. *The ring $\mathcal{A}$ can be expressed as*

$$\mathcal{A} = \{Y \in \mathcal{M} : \mu(Y) < \infty\}.$$

Proof. We only need to prove the "$\supseteq$" direction. Let $Y \in \mathcal{M}$ and $\mu(Y) < \infty$. one can find an open subset $V \supseteq Y$ such that $\mu(V) < \infty$ (thus $V \in \mathcal{A}$). Given $\varepsilon > 0$, let $K \subseteq V$ be a compact subset such that $\mu(V \backslash K) < \varepsilon$. Since $Y \cap K \in \mathcal{A}$, there exists a compact subset $K' \subseteq Y \cap K$ such that

$$\mu(Y \cap K) < \mu(K') + \varepsilon.$$

It follows that

$$\begin{aligned}\mu(Y) &= \mu(Y \cap K) + \mu(Y \cap K^c)\\ &< \mu(K') + \varepsilon + \mu(V \backslash K)\\ &< \mu(K') + 2\varepsilon.\end{aligned}$$

This shows that $Y \in \mathcal{A}$. □

The following lemma is the main result of this step.

Lemma 1.5. *The class $\mathcal{M}$ is a σ-algebra containing $\mathcal{B}(X)$, and the restriction of μ on $\mathcal{M}$ is a Radon measure.*

Proof. Since $\mathcal{A}$ contains all compact subsets, one knows that $X \in \mathcal{M}$. If $A \in \mathcal{M}$ and K is compact, then

$$A^c \cap K = K \backslash (A \cap K) \in \mathcal{A}$$

since $\mathcal{A}$ is a ring. Therefore, $A^c \in \mathcal{M}$. Finally, let $\{A_n : n \geqslant 1\} \subseteq \mathcal{M}$ and K be compact so that $B_n \triangleq A_n \cap K \in \mathcal{A}$ for each n. Note that $\cup_n B_n \subseteq K$ and thus $\mu(\cup_n B_n) < \infty$. By writing $\cup_n B_n$ as a disjoint union and using Lemma 1.3 (iii), (iv), one concludes that

$$\cup_n B_n = \big(\cup_n A_n\big) \cap K \in \mathcal{A}.$$

As a result, one sees that $\cup_n A_n \in \mathcal{M}$. Therefore, $\mathcal{M}$ is a σ-algebra. It is easy to see that $\mathcal{M}$ contains all closed subsets, and thus containing $\mathcal{B}(X)$.

To see that $\mu|_{\mathcal{M}}$ is a measure, let $\{A_n : n \geqslant 1\}$ be a disjoint sequence in $\mathcal{M}$. There is nothing to prove if $\mu(A_n) = \infty$ for some n. We may thus assume that $\mu(A_n) < \infty$ for all n. But this implies $A_n \in \mathcal{A}$ for all n according to Lemma 1.4. The countable additivity property thus follows from Lemma 1.3 (iii). The fact that μ is a Radon measure is clear from the definition of μ and Remark 1.5. □

1.2.3 Step three: showing that the functional λ coincides with the μ-integration

The last step for proving Theorem 1.1 is to show that

$$\lambda(f) = \int_X f d\mu \quad \forall f \in \mathcal{C}_c(X). \tag{1.6}$$

We first prepare two preliminary results.

Lemma 1.6. *Let K be a compact subset and $f \in \mathcal{C}_c(X)$ be such that $K \prec f$. Then $\mu(K) \leqslant \lambda(f)$.*

Proof. Let $\varepsilon > 0$. Pick an open subset $W \supseteq K$ such that $f \geqslant 1 - \varepsilon$ on W. Then pick $g \prec W$ such that

$$\mu(W) < \lambda(g) + \varepsilon.$$

Since $g \leqslant \frac{f}{1-\varepsilon}$, one sees that

$$\mu(K) \leqslant \mu(W) < \lambda(g) + \varepsilon \leqslant \frac{\lambda(f)}{1-\varepsilon} + \varepsilon.$$

The result follows by sending $\varepsilon \to 0$. $\square$

The next result is known as the *partition of unity*, which is often useful in the global analysis on manifolds/topological spaces.

Proposition 1.3. *Let K be a compact subset and let $\{U_1, \cdots, U_n\}$ be a finite open cover of K. Then there exists $f_i \prec U_i$ for each $1 \leqslant i \leqslant n$, such that*

$$K \prec f_1 + \cdots + f_n.$$

Proof. For each $x \in K$, there exists an open neighbourhood W_x of x such that $\overline{W_x} \subseteq U_i$ for some i (depending on x). By compactness, one can cover K by finitely many of the W_x's, say $W_{x_1}, \cdots, W_{x_m}$. For each $1 \leqslant i \leqslant n$, let V_i be the union of those W_{x_r}'s such that $\overline{W}_{x_r} \subseteq U_i$. It is clear that $\overline{V_i} \subseteq U_i$ and $\{\overline{V}_1, \cdots, \overline{V}_n\}$ covers K. Pick a function $\overline{V}_i \prec g_i \prec U_i$. We define

$$f_1 = g_1, \;\; f_2 = g_2(1 - g_1), \;\; f_3 = g_3(1 - g_1)(1 - g_2), \cdots$$

and

$$f_n = g_n(1 - g_1) \cdots (1 - g_{n-1}).$$

Then each $f_i \prec U_i$ and

$$f_1 + \cdots + f_n = 1 - (1 - g_1) \cdots (1 - g_n).$$

Note that on K at least one of the g_i's equals one. Therefore,

$$f_1 + \cdots + f_n = 1 \quad \text{on } K.$$

It is clear that $0 \leqslant \sum_{i=1}^{n} f_i \leqslant 1$. As a result, one has

$$K \prec f_1 + \cdots + f_n.$$

$\square$

Now we can develop the main proof for this step.

Lemma 1.7. *Let μ be the Radon measure constructed in Section 1.2.2. Then the identity (1.6) holds.*

Proof. By considering $-f$, it is enough to show that

$$\lambda(f) \leqslant \int_X f d\mu, \quad \forall f \in \mathcal{C}_c(X). \tag{1.7}$$

Given such f, let $K \triangleq \text{supp} f$. We first construct a standard step-function approximation of f. Suppose that $f(K) \subseteq (a, b)$. Let $\varepsilon > 0$ be a given number. We partition $[a, b]$ into

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that $a_i - a_{i-1} < \varepsilon$ for all i. Let $A_i \triangleq K \cap f^{-1}([a_{i-1}, a_i))$. Then the A_i's are disjoint and

$$K = \cup_{i=1}^{n} A_i.$$

Let

$$\varphi(x) \triangleq \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}.$$

It follows that

$$0 \leqslant \varphi - f \leqslant \varepsilon \quad \text{on } X.$$

Next, let $\Lambda \triangleq \max\{|a_i| : 0 \leqslant i \leqslant n\}$. For each i, we choose an open subset $V_i \supseteq A_i$ such that $f(x) < a_i$ for all $x \in V_i$ and

$$\mu(V_i) < \mu(A_i) + \frac{\varepsilon}{n\Lambda}.$$

This is possible by the continuity of f and the definition of $\mu(A_i)$. The open subsets $\{V_1, \cdots, V_n\}$ yield an open cover of K.

Now we use the partition of unity (cf. Proposition 1.3) to find $h_i \prec V_i$ ($1 \leqslant i \leqslant n$) such that

$$K \prec \sum_{i=1}^{n} h_i.$$

Note that $f = \sum_{i=1}^{n} fh_i$ and $fh_i \leqslant a_i h_i$ for each i. Therefore, one has

$$\begin{aligned}
\lambda(f) &= \sum_{i=1}^{n} \lambda(fh_i) \leqslant \sum_{i=1}^{n} a_i \lambda(h_i) = \sum_{i=1}^{n} (a_i + \Lambda)\lambda(h_i) - \Lambda \cdot \lambda\big(\sum_{i=1}^{n} h_i\big) \\
&\leqslant \sum_{i=1}^{n} (a_i + \Lambda)\mu(V_i) - \Lambda\mu(K) \quad \text{(by Lemma 1.6)} \\
&\leqslant \sum_{i=1}^{n} (a_i + \Lambda)\big(\mu(A_i) + \frac{\varepsilon}{n\Lambda}\big) - \Lambda\mu(K) \\
&\leqslant \sum_{i=1}^{n} a_i \mu(A_i) + \Lambda\mu(K) + 2\varepsilon - \Lambda\mu(K) \\
&= \int_X \varphi d\mu + 2\varepsilon \\
&\leqslant \int_X f d\mu + \varepsilon\mu(K) + 2\varepsilon.
\end{aligned}$$

The inequality (1.7) thus follows by letting $\varepsilon \to 0$. □

1.3 Indefinite integrals of Radon measures

We conclude this section by proving the following useful property of Radon measures. Let X be a locally compact, σ-compact Hausdorff space.

Proposition 1.4. *Let μ be a Radon measure over X and let f be a non-negative continuous function on X. Define the measure*

$$\nu(A) \triangleq \int_A f d\mu, \quad A \in \mathcal{B}(X).$$

Then ν is also a Radon measure.

Proof. It is clear that ν is a measure on $\mathcal{B}(X)$. We now check the Radon property.

(i) *Finiteness on compact subsets.* This is clear since μ is finite on compact sets and f is continuous (hence bounded on compact sets.)
(ii) *Inner regularity.* Let V be a given open subset. We only consider the case when $\int_V f d\mu < \infty$ as the other case requires only minor modification. Since X is σ-compact, one can find an sequence W_n of open subsets, such that

$$W_n \uparrow X, \ \overline{W}_n \text{ compact for each } n.$$

By the monotone convergence theorem, one has

$$\lim_{n\to\infty} \int_{V\cap W_n} f d\mu = \nu(V).$$

In particular, given $\varepsilon > 0$, there exists n such that

$$\nu(V) < \int_{V\cap W_n} f d\mu + \varepsilon.$$

Since μ is a Radon measure, one can find a compact subset $K \subseteq V \cap W_n$ such that

$$\mu\big(V \cap W_n \backslash K\big) < \frac{\varepsilon}{\|f\|_{\infty;\overline{W}_n}}.$$

It follows that

$$\nu(V) < \int_K f d\mu + \int_{V\cap W_n\backslash K} f d\mu + \varepsilon < \nu(K) + 2\varepsilon.$$

This proves the inner regularity of ν.
(iii) *Outer regularity.* Let $A \in \mathcal{B}(X)$. We assume that $\nu(A) < \infty$ for otherwise there is nothing to prove. Since X is σ-compact, one can write X as a disjoint union

$$X = \cup_{n=1}^{\infty} X_n,$$

where each X_n is contained in some open subset W_n with $\overline{W}_n$ compact. It follows that

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A \cap X_n) = \sum_{n=1}^{\infty} \int_{A\cap X_n} f d\mu.$$

Since μ is a Radon measure, for each n one finds an open subset V_n such that $A \cap X_n \subseteq V_n \subseteq W_n$ and

$$\mu\big(V_n \backslash (A \cap X_n)\big) < \frac{\varepsilon}{2^n \cdot \|f\|_{\infty;\overline{W}_n}}.$$

As a result, one has

$$\begin{aligned}
\nu(A) &= \sum_{n=1}^{\infty} \int_{V_n} f d\mu - \sum_{n=1}^{\infty} \int_{V_n \backslash (A \cap X_n)} f d\mu \\
&\geqslant \sum_{n=1}^{\infty} \nu(V_n) - \sum_{n=1}^{\infty} \|f\|_{\infty; \overline{W}_n} \cdot \mu\big(V_n \backslash (A \cap X_n)\big) \\
&\geqslant \nu\big(\cup_{n=1}^{\infty} V_n \big) - \varepsilon.
\end{aligned}$$

Since $\cup_{n=1}^{\infty} V_n$ is an open subset containing A, the outer regularity property follows.

$\square$

2 Haar measures and Haar functionals on locally compact Hausdorff groups

As an application of the RMK representation theorem, we construct invariant measures on a locally compact Hausdorff group, so that one can integrate functions on the group as well as on its factor spaces.

2.1 Some topological properties of locally compact Hausdorff groups

Let G be a given fixed locally compact Hausdorff group. Namely, G is a group, a locally compact Hausdorff space, and the multiplication and inversion

$$(x,y) \mapsto xy, \ x \mapsto x^{-1}$$

are both continuous operations. We collect a few basic definitions and topological properties of G, some of which are particularly elegant due to the presence of the group structure.

We use e to denote the identity of G. Given $x \in G$ and $A, B \subseteq G$, we write

$$xA \triangleq \{xa : a \in A\}, \ AB \triangleq \{ab : a \in A, b \in B\}.$$

A continuous function f on G is said to be *uniformly continuous*, if for any $\varepsilon > 0$, there exists an open neighbourhood V of e such that

$$x^{-1}y \in V \implies |f(y) - f(x)| < \varepsilon.$$

By making use the group multiplication, it is easy to describe the closure of a subset.

Lemma 2.1. *Let $A \subseteq G$. Then*

$$\bar{A} = \bigcap_V AV$$

where the intersection is taken over all open neighbourhoods of e.

Proof. Let $x \in \bar{A}$. Given any open neighbourhood V of e, one knows that xV^{-1} is a neighbourhood of x. Therefore, $xV^{-1} \cap A \neq \emptyset$. In other words, $xv^{-1} = y$ for some $v \in V$ and $y \in A$. It follows that

$$x = yv \in AV.$$

Since V is arbitrary, one co that $x \in \cap_V AV$. Conversely, let $x \in \cap_V AV$. Then for any open neighbourhood V of e, one has $xV^{-1} \cap A \neq \emptyset$. Since xV^{-1} is an arbitrary neighbourhood of x, one co that $x \in \bar{A}$. □

Lemma 2.2. *Let H be a subgroup of G. Then $\bar{H}$ is also a subgroup of G.*

Proof. (i) *$\bar{H}$ is closed under multiplication.* Let $h \in H$. Then

$$hH \subseteq H \subseteq \bar{H} \implies H \subseteq h^{-1}\bar{H} \implies \bar{H} \subseteq h^{-1}\bar{H} \implies h\bar{H} \subseteq \bar{H}.$$

This shows that $H\bar{H} \subseteq \bar{H}$. Next, let $x \in \bar{H}$. The previous fact gives

$$Hx \subseteq \bar{H} \implies H \subseteq \bar{H}x^{-1} \implies \bar{H} \subseteq \bar{H}x^{-1} \implies \bar{H}x \subseteq \bar{H}.$$

This implies that $\bar{H}\bar{H} \subseteq \bar{H}$.
(ii) *$\bar{H}$ is closed under inversion.* one has

$$H^{-1} = H \subseteq \bar{H} \implies H \subseteq \bar{H}^{-1} \implies \bar{H} \subseteq \bar{H}^{-1} \implies \bar{H}^{-1} \subseteq \bar{H}.$$

□

Next, we discuss some topological properties on factor spaces of G. Let H be a given subgroup of G. We define G/H to be the collection of left H-cosets:

$$G/H \triangleq \{xH : x \in G\}.$$

Let $\pi : G \to G/H$ be the canonical projection. The space G/H is equipped with the quotient topology, namely W is open in G/H if and only if $\pi^{-1}(W)$ is open in G. The following fact is quite useful in general.

Lemma 2.3. *(i) A subset $W \subseteq G/H$ is open if and only if $W = \pi(V)$ for some open subset $V \subseteq G$.*
(ii) A subset $L \subseteq G/H$ is compact if and only if $L = \pi(K)$ for some compact subset $K \subseteq G$.

Proof. (i) Let $W \subseteq G/H$ be open. Then $\pi^{-1}(W)$ is open in G and one also has $W = \pi(\pi^{-1}(W))$. Conversely, suppose that $W = \pi(V)$ with some open $V \subseteq G$. Then

$$\pi^{-1}(W) = \pi^{-1}(\pi V) = VH = \bigcup_{h \in H} Vh,$$

which is an open subset of G. Therefore, W is open in G/H.
(ii) Sufficiency is obvious. For the necessity, let L be a compact subset of G/H. Since G is locally compact, there exists a compact neighbourhood V of e (V open, $\bar{V}$ compact). From Part (i), one knows that $\pi(xV)$ is open in G/H for each $x \in G$. Note that $\{\pi(xV) : x \in G\}$ covers L. By compactness, one can find a finite sub-cover, say

$$L \subseteq \pi(x_1 V) \cup \cdots \cup \pi(x_n V).$$

We define

$$K \triangleq \pi^{-1}(L) \cap \big(x_1 \bar{V} \cup \cdots \cup x_n \bar{V}\big).$$

Then K is a compact subset of G, and it is routine to check that $\pi(K) = L$. $\square$

Note that G is a disjoint union of the left H-cosets xH. If H is an open subgroup, then xH is an open subset of G for each $x \in G$. As a result,

$$H^c = \bigcup_{x \notin H} xH$$

is a disjoint union of open subsets and is thus open. Equivalently, H is closed. This yields an interesting fact that every open subgroup of G is also closed. In particular, *if G is connected, the only open subgroup of G is the group G itself.*

On the other hand, closed subgroups are more important objects to consider.

Lemma 2.4. *A subgroup H is closed if and only if the factor space G/H is Hausdorff.*

Proof. Suppose that H is closed. Let $xH \neq yH$ (equivalently, $y^{-1}x \notin H$). Since H is closed, one can find an open neighbourhood V of e, such that $Vy^{-1}xV \subseteq H^c$ (consider the function $\varphi : G \times G \to G$ defined by $\varphi(z, w) \triangleq zy^{-1}xw$). We claim that

$$\pi(xV) \cap \pi(yV) = \emptyset. \tag{2.1}$$

Indeed, if it were not true, by definition one can find $v_1, v_2 \in V$ and $h_1, h_2 \in H$, such that $xv_1h_1 = yv_2h_2$. Equivalently, one has

$$v_2^{-1}y^{-1}xv_1 = h_2h_1^{-1} \in H,$$

which is a contradiction. Therefore, (2.1) holds and thus G/H is Hausdorff.

Conversely, suppose that G/H is Hausdorff. Let $x \notin H$. Then as elements in G/H one has $H \neq xH$. By assumption, one can pick two open neighbourhoods $W_1 \ni H$ and $W_2 \ni xH$ such that $W_1 \cap W_2 = \emptyset$. It follows that $\pi^{-1}(W_2)$ is an open neighbourhood of x and $\pi^{-1}(W_2) \subseteq H^c$. Therefore, H^c is open. $\square$

Lemma 2.4 gives a rather simple way of constructing non-Hausdorff spaces: one can take non-closed subgroups H and consider the corresponding factor spaces G/H. For instance, let $G = S^1$ and let H be the subgroup generated by $\sqrt{2}$. Then H is not closed and thus G/H is not Hausdorff.

We conclude with one more observation which will be used in the sequel.

Lemma 2.5. *Let H be a closed subgroup of G. The factor space G/H is locally compact and Hausdorff. In addition, if G is σ-compact, then G/H is also σ-compact.*

Proof. From Lemma 2.4 one knows that G/H is Hausdorff. To see local compactness, let $xH \in G/H$ be given. Since G is locally compact, one can find a compact neighbourhood V of x (V open, $\bar{V}$ compact). It follows that $xH \in \pi(V) \subseteq \pi(\bar{V})$. In other words, xH has a compact neighbourhood, and thus G/H is locally compact. If G is σ-compact, then $G = \cup_n K_n$ where K_n is an increasing sequence of compact subsets. It follows that $G/H = \cup_n \pi(K_n)$. In particular, G/H is also σ-compact. $\square$

2.2 Haar measures and Haar functionals

To integrate functions on G, one needs to have a suitable notion of measures. Since G is a locally compact Hausdorff group, a natural notion is Radon measures with certain G-invariant property.

Definition 2.1. A (*left*) *Haar measure* on G is a non-trivial Radon measure μ which is left invariant, in the sense that

$$\mu(xA) = \mu(A) \quad \forall x \in G,\ A \in \mathcal{B}(G).$$

If μ is a Haar measure, then all open subsets have positive μ-measures. Indeed, since μ is non-trivial, there exists $A \in \mathcal{B}(G)$ such that $\mu(A) > 0$. By outer and inner regularity, one can find a compact subset K such that $\mu(K) > 0$. Now suppose that U is an open subset with zero μ-measure. By left translation and compactness, one can cover K by $x_1U, \cdots, x_nU$ with $x_1, \cdots, x_n \in G$. Since μ is left-invariant, one has

$$\mu(K) \leqslant \sum_{i=1}^{n} \mu(x_iU) = n\mu(U) = 0,$$

which is a contradiction. Therefore, all open subsets have positive μ-measure.

In view of the connection between Radon measures and positive linear functionals, one also has the following definition.

Definition 2.2. A (*left*) *Haar functional* (also known as a (*left*) *Haar integral*) is a non-zero, positive linear function $\lambda : \mathcal{C}_c(G) \to \mathbb{C}$ which is left invariant, in the sense that

$$\lambda(l_x f) = \lambda(f) \quad \forall x \in G,\ f \in \mathcal{C}_c(G),$$

where $(l_x f)(y) \triangleq f(x^{-1}y)$.

By applying the RMK representation theorem (cf. Theorem 1.1) plus a little extra effort, one can obtain the following result.

Theorem 2.1. *There is a one-to-one correspondence between Haar measures and Haar functionals given by*

$$\mu \mapsto \lambda_\mu : \ \lambda_\mu(f) \triangleq \int_G f d\mu, \quad f \in \mathcal{C}_c(G).$$

Proof. The proof boils down to the following points.

(i) If μ is a Haar measure, then λ_μ is a Haar functional. Indeed, by approximating $f \in \mathcal{C}_c(G)$ by step functions, the left invariance of μ gives the left invariance of λ_μ:

$$\int_G f(ax)\mu(dx) = \int_G f(x)\mu(dx), \quad \forall a \in G.$$

(ii) Let λ be a Haar functional. According to the RMK representation theorem, there is a unique Radon measure μ such that $\lambda = \lambda_\mu$. According to left invariance of λ and the property (1.4), one sees that μ is left invariant on open subsets. It follows from outer regularity that μ is left invariant on $\mathcal{B}(G)$.

(iii) If μ, ν are two Haar measures giving the same integral $\lambda_\mu = \lambda_\nu$, then $\mu = \nu$. This is just the uniqueness part of the RMK representation theorem. □

We are now facing two natural questions: *Do Haar measures on G exist, and if yes how many are there*? The answer is contained in the following main result.

Theorem 2.2. *Haar measures on G exist, and they are unique up to the multiplication by positive scalars.*

We prove uniqueness and existence in the next two sections respectively. The approach we take relies on the functional viewpoint of Theorem 2.1.

2.3 The uniqueness of Haar measures

In order to prove the uniqueness part, we first consider the case when G is σ-compact. The main advantage of this extra assumption is that any Radon measure on G is σ-finite, and one can apply Fubini's theorem when evaluating double integrals.

Proof of the uniqueness part when G is σ-compact. Suppose that μ and ν are two Haar measures on G. Given a non-zero function $f \in \mathcal{C}_c(G)$, we define

$$\rho(f) \triangleq \frac{\int_G f d\mu}{\int_G f d\nu}.$$

Our goal is to show that $\rho(f)$ is a positive constant which is independent of f. This implies from Theorem 2.1 that μ and ν differ by a positive multiplicative scalar.

To this end, let W be a compact neighbourhood of e, and pick $h \prec W$. By considering $h(x)h(x^{-1})$ one may assume that $h(x) = h(x^{-1})$, and by normalisation one may also assume that

$$\int_G h(x)\nu(dx) = 1.$$

It follows that

$$\begin{aligned}
&\Big(\int_G h d\mu\Big) \cdot \Big(\int_G f d\nu\Big) - \Big(\int_G h d\nu\Big) \cdot \Big(\int_G f d\mu\Big) \\
&= \int_{G\times G} \big(h(x)f(y) - h(y)f(x)\big)\mu(dx)\nu(dy) \\
&= I - J,
\end{aligned}$$

where

$$\begin{aligned}
I &\triangleq \int_{G\times G} h(x)f(y)\mu(dx)\nu(dy) = \int_{G\times G} h(y^{-1}x)f(y)\mu(dx)\nu(dy) \\
&= \int_{G\times G} h(x^{-1}y)f(y)\mu(dx)\nu(dy) \overset{\text{Fubini}}{=} \int_G \mu(dx)\int_G h(x^{-1}y)f(y)\nu(dy) \\
&= \int_{G\times G} h(y)f(xy)\mu(dx)\nu(dy) \quad (x^{-1}y\mapsto y),
\end{aligned}$$

and

$$J \triangleq \int_{G\times G} h(y)f(x)\mu(dx)\nu(dy) = \int_{G\times G} h(y)f(yx)\mu(dx)\nu(dy) \quad (x\mapsto yx).$$

Therefore, one arrives at

$$\begin{aligned}
&\big(\int_G hd\mu\big)\cdot\big(\int_G fd\nu\big) - \big(\int_G hd\nu\big)\cdot\big(\int_G fd\mu\big) \\
&= \int_{G\times G} h(y)\big(f(xy)-f(yx)\big)\mu(dx)\nu(dy).
\end{aligned}$$

Since $f\in\mathcal{C}_c(G)$, given $\varepsilon>0$, when the neighbourhood W is small enough, the right hand side can be made smaller than $\varepsilon\cdot C_f$ where C_f is a constant depending on $\mu(\mathrm{supp} f)$ (the choice of h depends on W). As a result, one obtains that

$$\big|\int_G hd\mu - \rho(f)\big| \leqslant \frac{\varepsilon C_f}{\int_G fd\nu}.$$

In particular,

$$\rho(f) = \lim_{W\to\{e\}} \int_G hd\mu.$$

Since the limit does not depend on f, one co that $\rho(f)$ does not depend on f. The fact that $\rho(f)>0$ is obvious by taking $f>0$.

□

To get rid of the σ-compactness assumption, we rely on the following lemma.

Lemma 2.6. *There exists an open subgroup H of G which is σ-compact.*

Proof. Let K be a compact neighbourhood of e. By considering KK^{-1} one may assume that K is symmetric. Define $H\triangleq\cup_{n=1}^\infty K^n$. It is clear that H is a subgroup of G and H is σ-compact. To see that it is open, let $x\in H$. Then $x\in K^n$ for some n. It follows that $xK\subseteq K^{n+1}\subseteq H$. Therefore, H is open. □

Now we are able to prove the uniqueness part for the general case.

Proof of the uniqueness part without σ-compactness. Let H be a σ-compact open subgroup of G given by Lemma 2.6. Since the restrictions of μ and ν on H are Haar measures on H, one has essentially proven that, there exists a constant $c > 0$ such that

$$\int_G f d\nu = c \int_G f d\mu \quad \forall f \in \mathcal{C}_c(H).$$

Now write G as the disjoint union of left cosets xH. Let $f \in \mathcal{C}_c(G)$ and $K \triangleq \mathrm{supp} f$. By compactness, one can cover K by $x_1 H, \cdots, x_n H$ for some $x_1, \cdots, x_n$. Let $\{h_1, \cdots, h_n\}$ be a partition of unity subordinate to this open cover (cf. Proposition 1.3), i.e. $h_i \prec x_i H$ for each i and

$$K \prec h_1 + \cdots + h_n.$$

Then one has $l_{x_i^{-1}}(fh_i) \in \mathcal{C}_c(H)$ for each i. Therefore,

$$\begin{aligned}
\int_G f d\nu = \sum_{i=1}^n \int_G f h_i d\nu &= \sum_{i=1}^n \int_G l_{x_i^{-1}}(fh_i) d\nu \quad \text{(by left invariance)} \\
&= c \cdot \sum_{i=1}^n \int_G l_{x_i^{-1}}(fh_i) d\nu = c \cdot \int_G f d\mu.
\end{aligned}$$

□

2.4 The existence of Haar measures

We prove the existence of a Haar measure by constructing a Haar functional (cf. Theorem 2.1), i.e. a positive linear functional $\lambda : \mathcal{C}_c(G) \to \mathbb{R}$ which is non-zero and left-invariant. For this purpose, the main idea is to first construct a family of left-invariant functionals that are "almost additive", and then show that the "limiting object" obtained from this family is indeed additive. Here the perspective of taking limit is abstract but quite elegant: one thinks of this family of functionals as a collection of points in some compact topological space, and the existence of a "limit" for this collection will be a simple consequence of compactness. The approach we adopt here was due to H. Cartan [2].

The essential structure

Let L^+ denote the space of functions $f \in \mathcal{C}_c(G)$ such that $f \geqslant 0$. A major effort of the proof is to construct a family of functionals

$$\lambda_g : L^+ \to [0, \infty)$$

indexed by non-trivial elements $g \in L^+$, such that the following properties hold.

(i) λ_g is sub-additive:

$$\lambda_g(f_1 + f_2) \leqslant \lambda_g(f_1) + \lambda_g(f_2), \ \lambda_g(cf_1) = c\lambda_g(f_1) \tag{2.2}$$

for all $f_1, f_2 \in L^+$ and $c \geqslant 0$;
(ii) λ_g is "almost additive": given $f_1, f_2 \in L^+$ and $\varepsilon > 0$, there exists a neighbourhood V of e, such that

$$\lambda_g(f_1 + f_2) \geqslant \lambda_g(f_1) + \lambda_g(f_2) - \varepsilon \tag{2.3}$$

for all g whose support is contained in V;
(iii) λ_g is left-invariant, i.e.

$$\lambda_g(l_x f) = \lambda_g(f) \quad \forall f \in L^+ \text{ and } x \in G;$$

(iv) For each $f \in L^+$, there exist non-negative numbers $a_f \leqslant b_f$ (depending on f) such that $a_f > 0$ if $f \neq 0$, and

$$a_f \leqslant \lambda_g(f) \leqslant b_f \quad \forall g \in L^+, g \neq 0.$$

Presuming the existence of such a family $\{\lambda_g\}$, let us proceed to construct a desired Haar functional via certain topological limiting procedure. The precise construction of these λ_g's will be given in the next subsection.

Proof of the existence of Haar functionals. Let $\{\lambda_g\}$ be a family of functionals satisfying the above Properties (i) – (iv). We consider the product space

$$I \triangleq \prod_{f \in L^+} I_f, \quad I_f \triangleq [a_f, b_f]$$

equipped with the usual product topology. According to Tychonoff's theorem (cf. Theorem B.1), I is a compact topological space. By Property (iv), for each

$g \in L^+ \backslash \{0\}$, the functional λ_g can be equivalently viewed as a point in I. For each open neignbourhood V of e, we define S_V to be the closure of the point set

$$\{\lambda_g : \mathrm{supp} g \subseteq V\}$$

in I. Since

$$\{\lambda_g : \mathrm{supp} g \subseteq V_1 \cap \cdots \cap V_n\} \subseteq \{\lambda_g : \mathrm{supp} g \subseteq V_1\} \cap \cdots \cap \{\lambda_g : \mathrm{supp} g \subseteq V_n\},$$

it is clear that the family of compact subsets $S_V \subseteq I$ has the finite intersection property. Therefore, the intersection of all such S_V's is non-empty, say having a common element λ. Since $\lambda \in I$, it defines a functional

$$\lambda : L^+ \to \mathbb{R}, \ f \mapsto \pi_f(\lambda) \in I_f$$

in the obvious way.

We claim that λ is additive, left-invariant and non-zero. To see its additivity, let $f_1, f_2 \in L^+$ and $f_3 \triangleq f_1 + f_2$. Given $\varepsilon > 0$, let V be the neighbourhood of e such that Property (ii) holds. Since $\lambda \in S_V$, by considering the continuous projection

$$\pi^{(3)} : I \to I_{f_1} \times I_{f_2} \times I_{f_3},$$

one has

$$\pi^{(3)}(\lambda) \in \prod_{i=1}^{3} \overline{\{\lambda_g(f_i) : \mathrm{supp} g \subseteq V\}}.$$

As a result, there exists g such that $\mathrm{supp} g \subseteq V$ and

$$\big|\lambda(f_i) - \lambda_g(f_i)\big| < \varepsilon, \quad i = 1, 2, 3.$$

It follows from (2.2) and (2.3) that

$$\lambda(f_1) + \lambda(f_2) - 4\varepsilon \leqslant \lambda(f_1 + f_2) \leqslant \lambda(f_1) + \lambda(f_2) + 3\varepsilon.$$

Since ε is arbitrary, the additivity of λ thus follows. In a similar way, the left-invariance of λ is the consequence of the left-invariance of the λ_g's. Finally, since $\lambda(f) \geqslant a_f > 0$ whenever $f \neq 0$, one sees that λ is non-zero.

To conclude the proof, it remains to define the actual functional $\lambda : \mathcal{C}_c(G) \to \mathbb{R}$ by

$$\Lambda(f) \triangleq \lambda(f_1) - \lambda(f_2),$$

where $f = f_1 - f_2$ with $f_1, f_2 \in L^+$. The additivity of λ on L^+ shows that Λ is a well defined linear functional. The fact that Λ is a Haar functional is now obvious. $\square$

The construction of λ_g

We now proceed to give an explicit construction of the family $\{\lambda_g : L^+\backslash\{0\}\}$. To this end, we first make the following observation.

Lemma 2.7. *Let $f, g \in L^+$ and $g \neq 0$. Then there exist positive numbers $c_i > 0$ as well as $s_i \in G$ $(1 \leqslant i \leqslant n)$, such that*

$$f(x) \leqslant \sum_{i=1}^{n} c_i g(s_i x). \tag{2.4}$$

Proof. Let V be an open subset of G such that $g \geqslant m > 0$ on V. Since $K \triangleq \mathrm{supp} f$ is compact, one can cover K by $s_1 V, \cdots, s_n V$ with some $n \geqslant 1$ and $s_i \in G$ $(1 \leqslant i \leqslant n)$. Let

$$c_i \triangleq \frac{\sup f}{m}, \ i = 1, 2, \cdots, n.$$

Then the inequality (2.4) holds for such choices of the c_i, s_i's. □

Given $f, g \in L^+$, we define $(f : g)$ to be the infimum of all the sums $\sum_{i=1}^{n} c_i$ for all choices of $\{c_i, s_i\}$ satisfying (2.4). If $g = 0$, we take the convention that $(f : g) \triangleq \infty$. When $g \neq 0$, one has $(0 : g) = 0$ and from Lemma 2.7 one also knows that $(f : g) < \infty$.

Remark 2.1. From the proof of Lemma 2.7, one actually has (when $g \neq 0$)

$$(f : g) \leqslant \frac{n \sup f}{m}$$

where n, m are the numbers introduced in that proof. This inequality will be useful later on.

The basic properties of $(f : g)$ are summarised in the following lemma. The proof is almost immediate from the definition and is thus left as an exercise.

Lemma 2.8. *The symbol $(f : g)$ satisfies the following properties:*

(i) Left-invariance:

$$(l_x f : g) = (f : g) \quad \forall x \in G;$$

(ii) Sub-additivity:

$$(f_1 + f_2 : g) \leqslant (f_1 : g) + (f_2 : g);$$

(iii) Scaling:

$$(cf : g) = c(f : g) \quad \forall c \geqslant 0;$$

(iv) Monotonicity:

$$f_1 \leqslant f_2 \implies (f_1 : g) \leqslant (f_2 : g);$$

(v) For any $f, g, h \in L^+$, one has

$$(f : g) \leqslant (f : h)(h : g); \tag{2.5}$$

(vi) For any $f, g \in L^+$ not both being zero, one has

$$(f : g) \geqslant \frac{\sup f}{\sup g}.$$

Now let $h_0 \in L^+$ be a non-zero function that is fixed throughout the rest. Given $g \in L^+ \backslash \{0\}$, we define

$$\lambda_g(f) \triangleq \frac{(f : g)}{(h_0 : g)}, \quad f \in L^+.$$

It remains to show that the family $\{\lambda_g : g \in L^+ \backslash \{0\}\}$ satisfies the Properties (i) – (iv) in the last subsection. Properties (i) and (iii) are plain. Property (iv) follows from (2.5):

$$a_f := \frac{1}{(h_0 : f)} \leqslant \lambda_g(f) \leqslant (f : h_0) =: b_f. \tag{2.6}$$

It is clear that $a_f > 0$ when $f \neq 0$. The justification of Property (iii) is contained in the following lemma, after which the proof of the existence of Haar measures will be complete.

Lemma 2.9. *Let $f_1, f_2 \in L^+$ and $\varepsilon > 0$. Then there exists a neighbourhood V of e, such that*

$$\lambda_g(f_1 + f_2) \geqslant \lambda_g(f_1) + \lambda_g(f_2) - \varepsilon$$

for any g with $\mathrm{supp} g \subseteq V$.

Proof. Let h be such that $\mathrm{supp} f_1 \cup \mathrm{supp} f_2 \prec h$ (cf. (1.3)). Let $\delta > 0$ which will be chosen later on (depending on f_1, f_2 and ε). We define

$$f = f_1 + f_2 + \delta h,$$

and let $h_i \triangleq f_i/f$ $(i = 1, 2)$. Due to the presence of h, h_i is a well defined function in L^+. In particular, by uniform continuity there exists an open neighbourhood V of e such that

$$|h_i(y) - h_i(x)| < \delta \quad (i = 1, 2)$$

whenever $y \in xV$. Let g be an arbitrary non-zero function in L^+ such that supp$g \subseteq V$. Suppose that

$$f(x) \leqslant \sum_{j=1}^{n} c_j g(s_j x) \quad \forall x \in G \tag{2.7}$$

with some $\{c_j, s_j\}$. If $g(s_j x) \neq 0$, then $s_j x \in V$ or equivalently $x \in s_j^{-1}V$. It follows that

$$\begin{aligned} f_i(x) = f(x)h_i(x) &\leqslant \sum_{j=1}^{n} c_j g(s_j x) h_i(x) \\ &\leqslant \sum_{j=1}^{n} c_j g(s_j x)\big(h_i(s_j^{-1}) + \delta\big) \end{aligned}$$

for $i = 1, 2$. By definition one has

$$(f_i : g) \leqslant \sum_{j=1}^{n} c_j \big(h_i(s_j^{-1}) + \delta\big), \quad i = 1, 2.$$

Since $h_1 + h_2 \leqslant 1$, one obtains

$$(f_1 : g) + (f_2 : g) \leqslant (1 + 2\delta) \sum_{j=1}^{n} c_j,$$

and by taking infimum of the sums $\sum_{j=1}^{n} c_j$ satisfying (2.7), one has

$$\begin{aligned} &(f_1 : g) + (f_2 : g) \\ &\leqslant (1 + 2\delta)(f : g) \\ &\leqslant (1 + 2\delta)(f_1 + f_2 : g) + (1 + 2\delta)\delta(h : g). \end{aligned}$$

Dividing this by $(h_0 : g)$, one arrives at

$$\lambda_g(f_1) + \lambda_g(f_2) \leqslant \lambda_g(f_1 + f_2) + 2\delta\lambda_g(f_1 + f_2) + (1 + 2\delta)\delta\lambda_g(h).$$

The next observation is that (cf. (2.6))

$$\lambda_g(f_1 + f_2) \leqslant (f_1 + f_2 : h_0), \;\; \lambda_g(h) \leqslant (h : h_0),$$

both of which are further bounded by a constant depending only on f_1, f_2 and h_0 (cf. Remark 2.1), say $M = M_{f_1, f_2, h_0}$. The result thus follows by choosing δ such that

$$\big(2\delta + (1 + 2\delta)\delta\big)M < \varepsilon.$$

$\square$

2.5 The modular function

It is conventional to use left invariance for the definition of Haar measures, which is more consistent with the study of group actions. One can equivalently use right invariance to formulate Haar measures and Haar integrals. However, in general one cannot expect that Haar measures are both left and right invariant. A way of capturing the difference is through the so-called modular function.

To define the modular function, let μ be a given (left invariant) Haar measure. First note that, for any $a \in G$, the functional $\lambda^a : \mathcal{C}_c(G) \to \mathbb{C}$ defined by

$$\lambda^a(f) \triangleq \int_G f(xa^{-1})\mu(dx), \quad f \in \mathcal{C}_c(G)$$

is a non-trivial, left invariant, positive linear functional (i.e. a Haar functional). According to Theorem 2.2, there exists a unique real number $\Delta(a) > 0$ such that

$$\lambda^a(f) = \Delta(a) \int_G f(x)\mu(dx) \quad \forall f \in \mathcal{C}_c(G).$$

This defines a function $\Delta : G \to \mathbb{R}^* \triangleq (0, \infty)$. More explicitly, one has

$$\Delta(a) = \frac{\int_G f(xa^{-1})\mu(dx)}{\int_G f(x)\mu(dx)}, \quad a \in G, \tag{2.8}$$

where f is any given function in $\mathcal{C}_c(G)$ with $\int_G f d\mu \neq 0$. Since Haar measures are unique up to a multiplicative constant, the function Δ does not depend on the choice of the Haar measure μ.

Definition 2.3. The function $\Delta : G \to \mathbb{R}^*$ is called the *modular function* of G.

Proposition 2.1. *The modular function Δ is a continuous homomorphism.*

Proof. By the definition of Δ, for any $a, b \in G$ one has

$$\begin{aligned} \Delta(ab) \int_G f(x)\mu(dx) &= \int_G f(xb^{-1}a^{-1})\mu(dx) = \int_G (r_a f)(xb^{-1})\mu(dx) \\ &= \Delta(b) \int_G r_a f(x)\mu(dx) = \Delta(b)\Delta(a) \int_G f(x)\mu(dx). \end{aligned}$$

Therefore, Δ is a homomorphism. The continuity of Δ follows from the formula (2.8), as the integral in the numerator is continuous in a. $\square$

The following property gives a relation between left and right Haar measures through the modular function.

Proposition 2.2. *Let μ be a left Haar measure on G. Then the following statements hold true.*

(i) For any $f \in \mathcal{C}_c(G)$, one has

$$\int_G f(x^{-1})\Delta(x^{-1})\mu(dx) = \int_G f(x)\mu(dx).$$

(ii) If G is also σ-compact, then $\Delta(x^{-1})\mu(dx)$ is a right Haar measure.

Proof. (i) Define the functional

$$\lambda_\Delta(f) \triangleq \int_G f(x^{-1})\Delta(x^{-1})\mu(dx), \quad f \in \mathcal{C}_c(G).$$

It is clear that λ_Δ is a non-trivial, positive linear functional. We claim that λ_Δ is left invariant. Indeed, for $a \in G$, one has

$$\begin{aligned}
\lambda_\Delta(l_a f) &= \int_G (l_a f)(x^{-1})\Delta(x^{-1})\mu(dx) = \int_G f((xa)^{-1})\Delta(x^{-1})\mu(dx) \\
&= \Delta(a) \int_G f((xa)^{-1})\Delta((xa)^{-1})\mu(dx) \\
&= \Delta(a)\Delta(a^{-1}) \int_G f(x^{-1})\Delta(x^{-1})\mu(dx) \\
&= \lambda_\Delta(f),
\end{aligned}$$

where to reach the second last identity one has used (2.8) for the function $g(x) \triangleq f(x^{-1})\Delta(x^{-1})$. Therefore, λ_Δ is a left Haar functional. According to Theorem 2.2, there exists $C > 0$ such that

$$\lambda_\Delta(f) = C \int_G f(x)\mu(dx) \quad \forall f \in \mathcal{C}_c(G).$$

We now show that $C = 1$. To this end, given $\varepsilon > 0$, pick a compact neighbourhood W of e such that

$$|\Delta(x^{-1}) - 1| < \varepsilon \quad \forall x \in W,$$

and pick $f \prec W$ such that

$$f(x) = f(x^{-1}), \ \int_G f(x)\mu(dx) = 1.$$

It follows that

$$\begin{aligned}C &= \int_G f(x^{-1})\Delta(x^{-1})\mu(dx) \\ &= \int_G f(x)\Delta(x^{-1})\mu(dx) \\ &= 1 + \int_W f(x)\big(\Delta(x^{-1}) - 1\big)\mu(dx).\end{aligned}$$

Therefore,

$$|C-1| \leqslant \varepsilon \cdot \int_W f(x)\mu(dx) = \varepsilon.$$

Since ε is arbitrary, C has to be equal to one. As a result, one co that

$$\lambda_\Delta(f) = \int_G f(x)\mu(dx).$$

(ii) Since G is σ-compact, from Proposition 1.4 one knows that $\Delta(x^{-1})\mu(dx)$ is a Radon measure. Now consider the functional

$$\Lambda(f) \triangleq \int_G f(x)\Delta(x^{-1})\mu(dx), \quad f \in \mathcal{C}_c(G).$$

Then Λ is a non-trivial, positive linear functional. We claim that Λ is right invariant. Indeed, for any $a \in G$, according to Part (i) one has

$$\begin{aligned}\Lambda(r_a f) &= \int_G f(xa^{-1})\Delta(x^{-1})\mu(dx) = \int_G \hat{f}(ax^{-1})\Delta(x^{-1})\mu(dx) \\ &= \int_G \hat{f}(ax)\mu(dx) = \int_G \hat{f}(x)\mu(dx) = \int_G \hat{f}(x^{-1})\Delta(x^{-1})\mu(dx) \\ &= \int_G f(x)\Delta(x^{-1})\mu(dx) = \Lambda(f).\end{aligned}$$

According to Theorem (2.2) (more precisely, the right invariant version of the theorem), one co that $\Delta(x^{-1})\mu(dx)$ is a right Haar measure. $\square$

Proposition 2.2 suggests that left and right Haar measures can be different in general. There is a situation where left and right Haar measures are identical.

Proposition 2.3. $\Delta \equiv 1$ *if and only if all left Haar measures on* G *are right invariant.*

Proof. Let μ be a left Haar measure. According to (2.8), one has

$$\Delta \equiv 1 \iff \int_G f(xa)\mu(dx) = \int_G f(x)\mu(dx) \quad \forall a \text{ and } f \iff \mu \text{ is right invariant.}$$

□

Definition 2.4. A locally compact Hausdorff group G is called *unimodular* if $\Delta \equiv 1$, namely, if all left Haar measures are right invariant.

It is trivial that abelian groups are unimodular. Another situation of unimodular groups is the following.

Proposition 2.4. *All compact Hausdorff groups are unimodular.*

Proof. Let G be a compact Hausdorff group. The essential observation is that any continuous homomorphism $\psi : G \to \mathbb{R}^*$ has to be trivial. For if not, say

$$\sup_{x\in G} \psi(x) = r > 1.$$

Since G is compact, one finds $x_0 \in G$ such that $\psi(x_0) = r$. Then $\psi(x_0^2) = r^2 > r$ which is a contradiction. Therefore, as a particular continuous homomorphism the modular function Δ is trivial. □

There are non-compact unimodular groups, as seen from the following example.

Example 2.1. Let $G = \mathrm{GL}(n;\mathbb{R})$. One views G as an open subset of $\mathbb{R}^{n^2}$ in the obvious way, and let dx denote the Lebesgue measure. Then $\mu_L(dx) = \frac{dx}{|\mathrm{det} x|^n}$ is a left Haar measure on G. Indeed, for any $a \in G$ one has

$$\int_G f(ax)\frac{dx}{|\det x|^n} = \int_G f(y)\frac{|\det a^{-1}|^n dy}{|\det a^{-1}y|^n} = \int_G f(y)\frac{dy}{|\det y|^n}.$$

The measure μ_L is also right invariance by essentially the same line of calculation. In particular, $d^\times x \triangleq \frac{dx}{|x|}$ is a Haar measure on $\mathbb{R}^\times \triangleq \mathbb{R}\backslash\{0\}$.

Below is an example of a group that is not unimodular.

Example 2.2. Let G be the group of affine transformations $z \mapsto x_1 z + x_2$ where $x_1 \in \mathbb{R}^\times$ and $x_2 \in \mathbb{R}$. Explicitly, one has

$$G = \left\{ x = \begin{pmatrix} x_1 & x_2 \\ 0 & 1 \end{pmatrix} : x_1 \in \mathbb{R}^\times, x_2 \in \mathbb{R} \right\}$$

equipped with the usual matrix multiplication.

It is natural to look for left Haar measures of the form

$$\mu_L = \rho(x_1, x_2)dx_1dx_2,$$

where dx_1, dx_2 are the Lebesgue measures on $\mathbb{R}^\times, \mathbb{R}$ respectively, and $\rho(x_1, x_2)$ is some function to be determined. Given $a, y \in G$, note that

$$a^{-1}y = \begin{pmatrix} \frac{y_1}{a_1} & \frac{y_2}{a_1} - \frac{a_2}{a_1} \\ 0 & 1 \end{pmatrix}.$$

By a change of variables and the left invariance, for any $a = \begin{pmatrix} a_1 & a_2 \\ 0 & 1 \end{pmatrix} \in G$ and $f \in \mathcal{C}_c(G)$ one has

$$\int_G f(ax)\rho(x)dx_1dx_2 = \int_G f(y)\rho(a^{-1}y)\frac{dy_1dy_2}{a_1^2} = \int_G f(y)\rho(y)dy_1dy_2.$$

This suggests that

$$\frac{1}{a_1^2}\rho(\frac{y_1}{a_1}, \frac{y_2}{a_1} - \frac{a_2}{a_1}) = \rho(y_1, y_2).$$

As a result, $\rho(y_1, y_2)$ should not depend on y_2, and its dependence on y_1 is clear from the above scaling property. More explicitly, one has $\rho(y_1, y_2) = \frac{1}{y_1^2}$. It is now readily checked that

$$\mu_L = \frac{dy_1dy_2}{y_1^2} = \frac{1}{|y_1|} \cdot d^\times y_1dy_2$$

is a left Haar measure on G, where $d^\times y_1 \triangleq \frac{dy_1}{|y_1|}$ is the canonical left Haar measure on $\mathbb{R}^\times$.

Next we compute the modular function of G. By using (2.8), one has

$$\begin{aligned}\Delta(a) &= \frac{\int_G f(xa^{-1})\mu_L(dx)}{\int_G f(x)\mu_L(dx)} = \frac{\int_G f(y)\rho(ya) \cdot |a_1|dy_1dy_2}{\int_G f(x)\mu_L(dx)} \\ &= \frac{1}{|a_1|} \cdot \frac{\int_G f(y)\rho(y)dy_1dy_2}{\int_G f(x)\mu_L(dx)} = \frac{1}{|a_1|}.\end{aligned}$$

It follows that a right Haar measure on G is given by

$$\Delta(x^{-1})\mu_L(dx) = |x_1| \cdot \frac{dx_1dx_2}{x_1^2} = d^\times x_1dx_2.$$

The left and right Haar measures on G are apparently different in this case.

2.6 Integration on factor spaces

Let G be a locally compact Hausdorff group and let H be a closed subgroup of G. From the viewpoint of group actions as well as for geometric reasons, it is important to consider integration on the factor space G/H.

Let μ_H be a given fixed left Haar measure on H. For $f \in \mathcal{C}_c(G)$, consider the function

$$F(u) \triangleq \int_H f(uy)\mu_H(dy), \quad u \in G.$$

It is obvious that F is continuous. F is constant on left H-cosets, since

$$u_2 = u_1 h \implies F(u_2) = \int_H f(u_1 hy)\mu_H(dy) = \int_H f(u_1 z)\mu_H(dz) = F(u_1).$$

As a result, there is a well defined function f_H on G/H such that $F = f_H \circ \pi$ where $\pi : G \to G/H$ is the canonical projection. By the definition of the quotient topology and the continuity of F, one sees that f_H is continuous. In addition, if $K \triangleq \mathrm{supp} f$, it is not hard to see that $f_H = 0$ on $\pi(K^c)$. Therefore, $f_H \in \mathcal{C}_c(G/H)$.

Proposition 2.5. *The map $\mathcal{C}_c(G) \ni f \mapsto f_H \in \mathcal{C}_c(G/H)$ a linear epimorphism.*

Proof. We only need to check surjectivity. Let $f' \in \mathcal{C}_c(G/H)$ with support $K' \subseteq G/H$. According to Lemma 2.3 (ii), there is a compact subset $K \subseteq G$ such that $\pi(K) = K'$. Pick $g \in \mathcal{C}_c(G)$ such that $g \geqslant 0$ on G and $g > 0$ on K. By the definition of g_H one has $g_H > 0$ on K'. Define

$$f(x) \triangleq g(x) \cdot \frac{f'(\pi(x))}{g_H(\pi(x))}, \quad x \in G.$$

Then $f \in \mathcal{C}_c(G)$ as g does. The surjectivity property follows from

$$f_H(\pi(x)) = \int_H f(xy)\mu_H(dy) = \int_H g(xy) \cdot \frac{f'(\pi(xy))}{g_H(\pi(xy))}\mu_H(dy) = f'(\pi(x)).$$

□

From now on, we further assume that G is σ-compact for the convenience of using Fubini's theorem. Let K be a closed subgroup of G. It is often the case that K is compact but we do not need this assumption here. From Lemma 2.5, one knows that G/K is locally compact, Hausdorff and σ-compact. Let Δ_G (respectively, Δ_K) denote the modular function of G (respectively, K). Let μ_G (respectively, μ_K) be a given fixed left Haar measure on G (respectively, on K). The following result gives the construction of left G-invariant measures on G/K.

Theorem 2.3. *Suppose that $\Delta_G|_K = \Delta_K$. Then there exists a unique Radon measure $\mu_{G/K}$ on G/K, such that*

$$\int_{G/K} f_K d\mu_{G/K} = \int_G f d\mu_G, \quad \forall f \in \mathcal{C}_c(G).$$

In addition, $\mu_{G/K}$ is left G-invariant.

Proof. Uniqueness is a consequence of the uniqueness part for the general RMK representation theorem. One can use the same theorem to prove existence, by considering the positive linear functional

$$\Lambda(\varphi) \triangleq \int_G f d\mu_G$$

on $\mathcal{C}_c(G/K)$, where $f \in \mathcal{C}_c(G)$ is such that $f_K = \varphi$ whose existence is guaranteed by Proposition 2.5. The only non-trivial part is to show that Λ is well defined. This is equivalent to the following statement:

$$f_K = 0 \implies \int_G f d\mu_G = 0.$$

To this end, given $f \in \mathcal{C}_c(G)$, let $\psi \in \mathcal{C}_c(G/K)$ be such that $\psi = 1$ on $\pi(\mathrm{supp} f)$ and choose $g \in \mathcal{C}_c(G)$ with $g_K = \psi$. Then one has

$$\begin{aligned}
0 &= \int_G g(x) f_K(\pi(x)) \mu_G(dx) = \int_G g(x) \big(\int_K f(xk) \mu_K(dk) \big) \mu_G(dx) \\
&= \int_K \mu_K(dk) \int_G g(x) f(xk) \mu_G(dx) \\
&= \int_K \mu_K(dk) \cdot \Delta_G(k^{-1}) \int_G g(xk^{-1}) f(x) \mu_G(dx) \\
&= \int_G f(x) \mu_G(dx) \int_K \Delta_K(k^{-1}) g(xk^{-1}) \mu_K(dk) \quad \text{(by assumption)} \\
&= \int_G f(x) \mu_G(dx) \int_K g(xk) \mu_K(dk) = \int_G f(x) g_K(\pi(x)) \mu_G(dx) \\
&= \int_G f(x) \psi(\pi(x)) \mu_G(dx) = \int_G f(x) \mu_G(dx).
\end{aligned}$$

This shows that Λ is well defined. The RMK theorem then gives the existence of $\mu_{G/K}$. To see its left invariance, observe that $l_a f_K = (l_a f)_K$ for any $f \in \mathcal{C}_c(G)$. Therefore,

$$\int_{G/K} l_a f_K d\mu_{G/K} = \int_{G/K} (l_a f)_K d\mu_{G/K} = \int_G l_a f d\mu_G = \int_G f d\mu_G = \int_{G/K} f_K d\mu_{G/K}.$$

$\square$

Remark 2.2. An important case of Theorem 2.3 is when G and K are both unimodular.

There is a situation when G/K admits a further decomposition that is of particular importance. Let us assume that both G and K are unimodular. Suppose that there is another closed subgroup P of G, such that the map

$$P \times K \to G, \ (p,k) \mapsto pk \tag{2.9}$$

is a topological homeomorphism. Then P is homeomorphic to G/K via $p \mapsto pK$. In addition, the measure μ_P on P induced by $\mu_{G/K}$ (cf. Theorem 2.3) is a left Haar measure on P. According to the same theorem,

$$\int_G f(x)\mu_G(dx) = \int_P \Big(\int_K f(pk)\mu_K(dk)\Big)\mu_P(dp).$$

Symbolically, one can write $d\mu_G = d\mu_P d\mu_K$ under the identification (2.9).

Now suppose further that there are unimodular, closed subgroups A, N of P such that A normalises N (i.e. $ana^{-1} \in N$ for all $a \in A$ and $n \in N$) and the map

$$A \times N \to P, \ (a,n) \mapsto an$$

is a topological homeomorphism. Then G admits a unique decomposition $G = ANK$. Let da, dn, dk be given left Haar measures on A, N, K respectively.

Proposition 2.6. *The measure $dadn$ induced by the map*

$$A \times N \to P, \ (a,n) \mapsto an$$

is a left Haar measure on P. Similarly, the measure $dadndk$ induced on G by the map

$$A \times N \times K \to G, \ (a,n,k) \mapsto ank$$

is a left Haar measure on G.

Proof. We first show that $dadn$ is a left Haar measure on P. The A-invariance follows from

$$\int_P f(a_1^{-1}an)dadn = \int_N dn \int_A f(a_1^{-1}an)da = \int_N dn \int_A f(an)da = \int_P f(an)dadn,$$

for each given $a_1 \in A$. The N-invariance follows from

$$\begin{aligned}\int_P f(n_1^{-1}an)dadn &= \int_A da \int_N f(aa^{-1}n_1^{-1}an)dn \\ &= \int_A da \int_N f(an)dn \quad (\text{since } a^{-1}n_1^{-1}a \in N) \\ &= \int_P f(an)dadn.\end{aligned}$$

Therefore, $dadn$ is left invariant on P.

On the other hand, if μ_G is a given Haar measure on G, one knows from Theorem 2.3 that $\mu_{G/K}$ is left G-invariant and its induced measure on P (still denoted as $\mu_{G/K}$) is a left Haar measure. In particular, $\mu_{G/K} = cdadn$ for some $c > 0$. Since $\mu_G = \mu_{G/K} \times dk$ via the identification $G/K \times K \approx G$, one co that

$$\mu_G = cdadndk.$$

In particular, $dadndk$ is a Haar measure on G. □

Let $dx \triangleq dadndk$. Since A normalises N, one also has $P = NA$ and thus $G = NAK$. There is also a measure $dndadk$ induced by the NAK-decomposition. To understand the relation between dx and $dndadk$, we need the following lemma.

Lemma 2.10. *There is a continuous homomorphism $\alpha : A \to \mathbb{R}^*$, such that*

$$\int_N f(an)dn = \alpha(a)^{-1} \int_N f(na)dn \tag{2.10}$$

for any $a \in A$ and $f \in \mathcal{C}_c(P)$. In addition, the modular function of P is given by

$$\Delta(p) = \alpha(a)^{-1}, \ p = an \in P.$$

Proof. Given $a \in A$, define

$$\Lambda_a(f) \triangleq \int_N f(na)dn, \quad f \in \mathcal{C}_c(P).$$

It is plain to check that $\Lambda_a(f)$ is a left Haar functional on $\mathcal{C}_c(P)$. By uniqueness one knows that there exists $\alpha(a) > 0$ such that

$$\Lambda_a(f) = \alpha(a) \int_N f(an)dn, \quad \forall f \in \mathcal{C}_c(P).$$

The continuity and homomorphism property of α follows from the same reason as for the case of the modular function Δ (cf. Proposition 2.1).

To prove the second part of the lemma, consider the left Haar functional

$$\Lambda(f) \triangleq \int_P f(an)dadn, \quad f \in \mathcal{C}_c(P).$$

Since N is unimodular, it is obvious that Λ is right N-invariant. In addition, given $a_1 \in A$, one has

$$\begin{aligned}\int_P f(ana_1)dadn &= \int_A \big(\int_N f(aa_1a_1^{-1}na_1)dn\big)da \\ &= \alpha(a_1)\int_A \big(\int_N f(aa_1n)dn\big)da \\ &= \alpha(a_1)\int_A \big(\int_N f(an)dn\big)da \quad \text{(since } A \text{ is unimodular)} \\ &= \alpha(a_1)\int_P f(an)dadn.\end{aligned}$$

By the definition of the modular function, one co that $\Delta(an) = \alpha(a)^{-1}$. $\square$

Remark 2.3. In general, P may not be unimodular (when the homomorphism α is non-trivial).

The relation between dx and $dndadk$ is given by the following result.

Proposition 2.7. *Let $\alpha : A \to \mathbb{R}^*$ be the homomorphism given by Lemma 2.10. Then*

$$dx = \alpha(a)^{-1}dndadk.$$

Proof. For $f \in \mathcal{C}_c(G)$, one has

$$\begin{aligned}\int_G f(x)dx = \int_{A\times N\times K} f(ank)dadndk &= \int_K dk\int_A da\int_N f(ank)dn \\ &= \int_K dk\int_A \alpha(a)^{-1}da\int_N f(nak)dn \\ &= \int_{N\times A\times K}\alpha(a)^{-1}f(nak)dndadk.\end{aligned}$$

The result thus follows. $\square$

Any of the following decompositions (the latter two are obtained by taking inverse of the former two)

$$G = ANK \text{ or } NAK \text{ or } KAN \text{ or } KNA$$

is known as the *Iwasawa decomposition* of G.

2.7 An example: the special linear group $\mathrm{SL}_2(\mathbb{R})$

We conclude by an enlightening example: the special linear group $\mathrm{SL}_2(\mathbb{R})$. Mathematically, it is defined by

$$\mathrm{SL}_2(\mathbb{R}) \triangleq \big\{g \in \mathrm{Mat}(2;\mathbb{R}) : \det g = 1\big\}.$$

2.7.1 $\mathrm{SL}_2(\mathbb{R})$-action via the Möbius transformation.

From a geometric viewpoint, $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H} = \{x+yi : y > 0\}$ by isometry, where the group action is defined by the Möbius transformation and $\mathbb{H}$ is equipped with the Lobachevsky hyperbolic metric.

To elaborate this, given $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z = x+yi \in \mathbb{H}$, we define

$$g(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = \frac{(\alpha\gamma|z|^2 + \beta\delta) + (\alpha\delta z + \beta\gamma\bar{z})}{|\gamma z + \delta|^2}.$$

Since $\det g = 1$, it follows that

$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|\gamma z + \delta|^2}. \tag{2.11}$$

In particular, g leaves $\mathbb{H}$ invariant.

Next, we try to figure out the *isotropy subgroup* at i, i.e. the subgroup of $\mathrm{SL}_2(\mathbb{R})$ leaving i fixed. Suppose that $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ fixes i. Then $\frac{\alpha i+\beta}{\gamma i+\delta} = i$, or equivalently

$$\alpha i + \beta = \delta i - \gamma.$$

It follows that

$$\alpha = \delta,\ \beta = -\gamma.$$

Note that $\alpha^2 + \gamma^2 = 1$ since $\det g = 1$. By writing $\alpha = \cos\theta, \gamma = \sin\theta$, one sees that g belongs to the circle subgroup

$$K = \left\{ k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}. \tag{2.12}$$

Conversely, it is obvious that any element in K leaves i fixed. Therefore, K is the isotropy subgroup at i. It follows that the coset space $\mathrm{SL}_2(\mathbb{R})/K$ is diffeomorphic to $\mathbb{H}$.

One can say more about this $\mathrm{SL}_2(\mathbb{R})$-action on $\mathbb{H}$. Recall that, when equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \tag{2.13}$$

the space $\mathbb{H}$ becomes a hyperbolic manifold (the *Lobachevsky plane*) of constant negative curvature -1.

Proposition 2.8. *Under the hyperbolic metric,* $\mathrm{SL}_2(\mathbb{R})$ *acts on* $\mathbb{H}$ *by isometry.*

Proof. Using complex coordinates $(z, \bar{z})$, the hyperbolic metric can be written as

$$ds^2 = \frac{dzd\bar{z} + d\bar{z}dz}{2(\mathrm{Im}z)^2}, \ z \in \mathbb{H}.$$

Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{R})$ and

$$w \triangleq g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Then

$$\frac{\partial w}{\partial z} = \frac{1}{(\gamma z + \delta)^2}, \ \frac{\partial \bar{w}}{\partial \bar{z}} = \frac{1}{(\gamma \bar{z} + \delta)^2}, \ \frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0.$$

It follows that

$$\begin{aligned} g^*(ds^2) &= \frac{dwd\bar{w} + d\bar{w}dw}{2(\mathrm{Im}w)^2} \\ &= \frac{\partial w}{\partial z} \cdot \frac{\partial \bar{w}}{\partial \bar{z}} \cdot \frac{dzd\bar{z} + d\bar{z}dz}{2(\mathrm{Im}w)^2} \\ &= \frac{dzd\bar{z} + d\bar{z}dz}{2(\mathrm{Im}w)^2 \cdot |\gamma z + \delta|^4} \\ &= \frac{dzd\bar{z} + d\bar{z}dz}{2(\mathrm{Im}z)^2}. \end{aligned}$$

To reach the last equality, one has used the relation (2.11). Therefore, $g^*(ds^2) = ds^2$ and thus g is an isometry. $\square$

2.7.2 The Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$

Although one can consider the ANK decomposition as before, from the geometric viewpoint it is more natural to consider the NAK decomposition. We first define these closed subgroups respectively:

$$A \triangleq \left\{a = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} : r > 0\right\}, \ N \triangleq \left\{n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$$

and K is the previous circle group defined by (2.12). Note that A normalises N, as seen from the following relation:

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^2x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \tag{2.14}$$

To write down the decomposition $\mathrm{SL}_2(\mathbb{R}) = NAK$ effectively, the main observation is that the group action $g \mapsto g(i)$ restricts to a diffeomorphism between the subgroup $P = NA$ and the upper half-plane $\mathbb{H}$. Indeed, plain calculation shows that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}(i) = x + r^2 i. \tag{2.15}$$

Now suppose that

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.16}$$

By using the relation (2.15), one has

$$g(i) = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + \frac{1}{\gamma^2 + \delta^2} i = x + r^2 i.$$

As a result,

$$x = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2}, \ r = \frac{1}{\sqrt{\gamma^2 + \delta^2}}. \tag{2.17}$$

To figure out θ, explicit multiplication shows that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} r\cos\theta - \frac{x}{r}\sin\theta & r\sin\theta + \frac{x}{r}\cos\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}.$$

By using the expression for r, one sees that

$$\cos\theta = \frac{\delta}{\sqrt{\gamma^2 + \delta^2}}, \ \sin\theta = -\frac{\gamma}{\sqrt{\gamma^2 + \delta^2}},$$

or more concisely,

$$e^{i\theta} = \frac{\delta - i\gamma}{\sqrt{\gamma^2 + \delta^2}} \in K. \tag{2.18}$$

The equations (2.17) and (2.18) gives the NAK decomposition of $\mathrm{SL}_2(\mathbb{R})$ in explicit form, which not only shows the existence of such decomposition but also gives uniqueness. By using the relation (2.14), it is easy to write down the ANK decomposition as well. Note that $P = NA = AN$.

2.7.3 Haar measures on $\mathrm{SL}_2(\mathbb{R})$

Before considering Haar measures, we first show that $\mathrm{SL}_2(\mathbb{R})$ is unimodular, namely, the modular function $\Delta \equiv 1$. Recall that Δ is a continuous homomorphism from $\mathrm{SL}_2(\mathbb{R})$ to $\mathbb{R}^*$. It is thus sufficient to establish the following more general fact.

Proposition 2.9. *There are no non-trivial continuous homomorphisms from* $\mathrm{SL}_2(\mathbb{R})$ *to* $\mathbb{R}^*$.

Proof. By taking logarithm, it suffices show that any continuous homomorphism from $\mathrm{SL}_2(\mathbb{R})$ to the additive group $(\mathbb{R}, +)$ is trivial. Let $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ be such a homomorphism. It follows from the Iwasawa decomposition that

$$\varphi(nak) = \varphi(n) + \varphi(a) + \varphi(k).$$

Therefore, it is enough to show that φ is trivial when restricted on the subgroups N, A, K respectively.

In the first place, K contains a dense subset K_0 of elements of finite order. Since every non-zero element in $\mathbb{R}$ has infinite order, one knows that $\varphi|_{K_0} = 0$. Therefore, $\varphi|_K = 0$. Next, we consider $\varphi|_A$. Let $a = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \in A$. Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot a \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = a^{-1},$$

one has

$$\varphi(a) = \varphi(a^{-1}) = -\varphi(a),$$

giving $\varphi(a) = 0$. Therefore, $\varphi|_A = 0$. Finally, to consider $\varphi|_N$ let us observe that N is isomorphic to $\mathbb{R}$ through the obvious identification

$$N \ni \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longleftrightarrow x \in \mathbb{R}.$$

Since any homomorphism over $\mathbb{R}$ is given by the multiplication by a real number, there exists $t \in \mathbb{R}$ such that

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = tx \quad \forall x \in \mathbb{R}.$$

On the other hand, by applying φ to the relation (2.14), one has

$$tx = tr^2 x \quad \forall r > 0 \text{ and } x \in \mathbb{R}.$$

Therefore, $t = 0$ and thus $\varphi|_N = 0$. $\square$

Since $\mathrm{SL}_2(\mathbb{R})$ is unimodular and K is clearly unimodular as a compact group, the previous general results apply to the current situation. In a canonical way, the Haar measures on A, N, K are chosen as

$$da = \frac{d^+r}{r}, \ dx, \ d\theta$$

respectively, where d^+r is the Lebesgue measure on $\mathbb{R}^*$, dx is the Lebesgue measure on $\mathbb{R}$ and $d\theta$ is the normalised uniform measure on S^1. It follows from Proposition 2.6 and Proposition 2.7 that

$$dx = dadndk = \alpha(a)^{-1}dndadk$$

defines a Haar measure on $\mathrm{SL}_2(\mathbb{R})$.

Lemma 2.11. *The homomorphism α is given by $\alpha(a) = r^2$ for $a = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \in A$.*

Proof. Let $a = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ be fixed and $f \in \mathcal{C}_c(P)$. Define

$$\varphi(b) \triangleq f\left(a \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right), \ \psi(c) \triangleq f\left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \cdot a\right), \quad b, c \in \mathbb{R}.$$

By using (2.14), one easily sees that $\varphi(b) = \psi(r^2 b)$. Therefore,

$$\int_N f(na)dn = \int_{\mathbb{R}} \psi(c)dc = r^2 \int_{\mathbb{R}} \varphi(b)db = r^2 \int_N f(an)dn.$$

According to (2.10), one co that $\alpha(a) = r^2$. $\square$

Finally, it is interesting to observe that under the diffeomorphism (2.15) between $P = NA$ and $\mathbb{H}$, the measure on $\mathbb{H}$ induced by $2\alpha(a)^{-1}dnda$ is precisely the Riemannian volume form $\frac{dxdy}{y^2}$ with respect to the hyperbolic metric ds^2 (cf. (2.13)). To see this, note that in terms of the relevant coordinates the map $P \to \mathbb{H}$ is given by

$$(x, r) \mapsto (x, y = r^2).$$

As a result, one has $dy = 2rd^+r$ and thus

$$\frac{dxdy}{y^2} = \frac{2rdxd^+r}{r^4} = \frac{2}{r^2}dx\frac{d^+r}{r} = 2\alpha(a)^{-1}dnda.$$

By Lemma 2.10, the above measure is also equal to $2dadn$ under the decomposition $P = AN$.

3 The Peter-Weyl Theorem and Fourier analysis on compact Hausdorff groups

The classical theory of Fourier series asserts that every element $f \in L^2(S^1, d\theta)$ (S^1 is the unit circle, $d\theta$ is the normalised Haar measure so that S^1 has unit volume) admits an L^2-decomposition

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\theta}, \tag{3.1}$$

where $c_k \triangleq \langle f, e^{ik\theta} \rangle_{L^2}$ is the k-th *Fourier coefficient* of f. The family $\{e^{ik\theta} : k \in \mathbb{Z}\}$ of continuous functions form an orthonormal basis (ONB) of $L^2(S^1, d\theta)$. The equation (3.1) can be viewed as the *Fourier inversion formula* in this context. The classical *Parseval's Theorem* asserts that

$$\|f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} |c_k|^2.$$

One can think of $\mathbb{Z}$ as the "spectrum" of S^1: the "frequencies" are given by the integers each occurring with multiplicity one.

There is an elegant counterpart of the above classical results in the context of compact Hausdorff groups. Let G be such a group with normalised Haar measure dx. In this case, the inversion formula (3.1) for $f \in L^2(G)$ takes the form

$$f(x) = \sum_{\pi} d_\pi \langle \hat{f}(\pi), \pi(x)^* \rangle_{\mathrm{HS}} \tag{3.2}$$

Here the summation is taken over all *unitary irreducible representations* π (they are all finite dimensional). The object $\hat{f}(\pi)$ is interpreted as the Fourier coefficient at π and d_π is the dimension of the representation π. Parseval's theorem in this situation becomes

$$\|f\|_{L^2}^2 = \sum_{\pi} d_\pi \|\hat{f}(\pi)\|_{\mathrm{HS}}^2, \tag{3.3}$$

which is better known as *Plancherel's theorem.* All the objects appearing in these identities will be made precise later on. In vague terms, the "spectrum" of G is indexed by unitary irreducible representations, and each "frequency" occurs with multiplicity given by the dimension of the underlying representation.

The purpose of this section is to give a self-contained discussion on these results, which were essentially due to F. Peter and H. Weyl in the 1920s. A substantial part of the theory is related to understanding the structure of unitary representations (the Peter-Weyl theorem).

3.1 Basic definitions

We begin with some general definitions about group representations. Let G be a locally compact Hausdorff group. All Banach and Hilbert spaces are assumed to be defined over $\mathbb{C}$.

Definition 3.1. Let H be a Banach space. A *representation* of G on H is a group homomorphism $\pi : G \to \mathrm{Aut}(H)$ (the group of *continuous* linear automorphisms on H) such that for every $v \in H$, the map $x \mapsto \pi(x)v$ is continuous. When H is a Hilbert space, a representation is said to be *unitary* if $\pi(x)$ is a unitary operator on H for every $x \in G$. A representation is said to be *finite dimensional* if H is finite dimensional.

Definition 3.2. Let $\pi : G \to \mathrm{Aut}(H)$ be a given representation. A *G-invariant subspace* is a closed subspace W of H such that $\pi(x)W \subseteq W$ for any $x \in G$. A representation π is *irreducible* if H does not possess G-invariance subspaces other than $\{0\}$ and H.

Definition 3.3. Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary representation of G on some Hilbert space H. We say that H is *completely reducible* for π, if H can be written as an orthogonal direct sum of non-trivial irreducible subspaces.

We often do not distinguish representations that are G-isomorphic. To make this precise, let us introduce the following definition.

Definition 3.4. Let $\pi_i : G \to \mathrm{Aut}(H_i)$ $(i = 1, 2)$ be two representations of G, and let $\Phi : H_1 \to H_2$ be a bounded linear operator. We say that Φ is a *G-homomorphism* if $\Phi \circ \pi_1(x) = \pi_2(x) \circ \Phi$ for all $x \in G$. It is a *G-isomorphism* if Φ is bijective.

From now on, we restrict ourselves to the case when G is a *compact* Hausdorff group. From Proposition 2.4, one knows that G is unimodular. We use dx to denote the normalised Haar measure on G, i.e. $\int_G dx = 1$. Unless otherwise stated, we only consider representations of G on *Hilbert spaces*.

The first observation in this case is that essentially one only needs to consider unitary representations. This is due to the following simple but useful principle of averaging, which is a nice consequence of compactness.

Lemma 3.1. *Let $\pi : G \to \mathrm{Aut}(H)$ be a representation of G on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Define a new inner product*

$$\langle v, w \rangle_G \triangleq \int_G \langle \pi(x)v, \pi(x)w \rangle dx, \quad v, w \in H.$$

Then $\langle\cdot,\cdot\rangle_G$ and $\langle\cdot,\cdot\rangle$ are equivalent. In addition, π is a unitary representation under $\langle\cdot,\cdot\rangle_G$, i.e.

$$\langle\pi(x)v,\pi(x)w\rangle_G=\langle v,w\rangle_G$$

for all $x\in G$ and $v,w\in H$.

Proof. For each $v\in H$, since $x\mapsto\pi(x)v$ is continuous, from the compactness of G one knows that

$$\sup_{x\in G}\|\pi(x)v\|<\infty.$$

It follows from the Uniform Boundedness Theorem (cf. Theorem C.1 in Appendix C) that

$$M\triangleq\sup_{x\in G}\|\pi(x)\|_{H\to H}<\infty.$$

As a result, one has

$$\|v\|_G^2=\int_G\|\pi(x)v\|^2dx\leqslant M^2\|v\|^2.$$

Conversely, one also has

$$\|v\|^2=\|\pi(x^{-1})\pi(x)v\|^2\leqslant M^2\|\pi(x)v\|^2\quad\forall x\in G.$$

After integration over G, one obtains

$$\|v\|^2\leqslant M^2\|v\|_G^2.$$

Therefore, the inner products $\langle\cdot,\cdot\rangle_G$ and $\langle\cdot,\cdot\rangle$ are equivalent. To show the unitarity of π under $\langle\cdot,\cdot\rangle_G$, one simply observes that

$$\begin{aligned}\langle\pi(x)v,\pi(x)w\rangle_G&=\int_G\langle\pi(y)\pi(x)v,\pi(y)\pi(x)w\rangle dy\\&=\int_G\langle\pi(yx)v,\pi(yx)w\rangle dy\\&=\int_G\langle\pi(z)v,\pi(z)w\rangle dz\quad\text{(right invariance of }dy)\\&=\langle v,w\rangle_G.\end{aligned}$$

□

As a result of Lemma 3.1, one can restrict to unitary representations in the context of Hilbert spaces.

There is a particularly important unitary representation known as the regular representation which we now describe. Let $L^2(G)$ be the space of complex-valued square integrable functions on G. $L^2(G)$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{L^2} \triangleq \int_G f(x)\overline{g(x)}dx.$$

For each $y \in G$, we define its action $T(y)$ on $L^2(G)$ by right translation, namely

$$(T(y)f)(x) \triangleq f(xy), \quad f \in L^2(G).$$

From the invariance of the Haar measure dx, it is routine to check that

$$T : G \to \mathrm{Aut}(L^2(G))$$

defines a unitary representation of G on $L^2(G)$.

Definition 3.5. The above representation T is called the *regular representation* of G.

Among other properties, the importance of the regular representation lies in the following two aspects:

(i) it is completely reducible;
(ii) all unitary irreducible representations of G are contained in the regular representation.

The precise formulation of this fact as well as other related properties is the content of the *Peter-Weyl theorem*, which will be elaborated in Section 3.3 below.

3.2 A complete reducibility theorem for compact operators

The reason of reducibility is closely related to the use of compact operators due to the spectral theorem (cf. Theorem C.3 in Appendix C). Here we derive a general fact about compact operators which will be used later on.

We first give a few more definitions. Let $\mathcal{A}$ be a family of bounded linear operators on a Hilbert space H.

Definition 3.6. We say that $\mathcal{A}$ *is* $*$*-closed*, if

$$A \in \mathcal{A} \implies A^* \in \mathcal{A}$$

where A^* denotes the adjoint of A (i.e. $\langle Av, w\rangle = \langle v, A^*w\rangle$). We say that $\mathcal{A}$ is an *algebra*, if

$$A, B \in \mathcal{A}, c \in \mathbb{C} \implies cA + B,\ AB \in \mathcal{A}.$$

Definition 3.7. A closed subspace W of H is said to be $\mathcal{A}$-*invariant*, if $AW \subseteq W$ for all $A \in \mathcal{A}$. We say that H is $\mathcal{A}$-*irreducible* if it does not contain $\mathcal{A}$-invariant subspaces other than $\{0\}$ and H. We say that H is *completely reducible for* $\mathcal{A}$, if H can be written as an orthogonal direct sum of non-trivial $\mathcal{A}$-irreducible subspaces:

$$H = \bigoplus_{i \in I} H_i, \tag{3.4}$$

where the right hand side of (3.4) is understood as the closure of the algebraic direct sum.

one has the following complete reducibility property for a $*$-closed algebra of compact operators.

Proposition 3.1. *Let $\mathcal{A}$ be a $*$-closed algebra of compact operators acting on a Hilbert space H. Then the following statements hold true.*

(i) H is completely reducible.
(ii) Let H_i be an $\mathcal{A}$-irreducible subspace appearing in the decomposition (3.4). If at least one member of $\mathcal{A}$ acts non-trivially on H_i, then H_i occurs for at most finitely many times in the decomposition (up to $\mathcal{A}$-isomorphism).

Proof. (i) The key step is to show that there exists a non-trivial $\mathcal{A}$-irreducible subspace of H. If this is true, the rest of the argument is a standard application of Zorn's lemma (cf. Theorem A.1 in Appendix A). Indeed, let $\mathcal{P}$ denote the set of families of $\mathcal{A}$-irreducible orthogonal subspaces of H. In other words, a generic element of $\mathcal{P}$ is a family $\mathcal{F} = \{H_i : i \in \mathcal{I}\}$ of subspaces where each H_i is irreducible and $H_i \perp H_j$ if $i \neq j \in \mathcal{I}$. We define a partial order on $\mathcal{P}$ by the natural inclusion. It can be checked that every totally ordered subset of $\mathcal{P}$ has an upper bound in $\mathcal{P}$. According to Zorn's lemma, there is a maximal element, say $\mathcal{F} = \{H_i : i \in \mathcal{I}\}$. Let $V \triangleq \oplus_{i\in\mathcal{I}} H_i$ (more precisely, the closure of the algebraic direct sum). We claim that $V = H$. If this were not true, then $V^\perp$ is a non-trivial $\mathcal{A}$-invariant subspace of H. From what we have presumed, one may find a non-trivial irreducible subspace W of $V^\perp$. But $\{W\} \cup \mathcal{F}$ is an element of $\mathcal{P}$ that is strictly larger than $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$. Therefore, $V = H$.

It remains to establish the existence of a non-trivial $\mathcal{A}$-irreducible subspace. One may assume that there is at least a non-zero element $A \in \mathcal{A}$ for otherwise the claim is trivial. Note that both of $B \triangleq i(A - A^*)$ and $C \triangleq A + A^*$ are self-adjoint, and at least one of them is non-zero. Therefore, one may find a non-zero self-adjoint element in $\mathcal{A}$ (still denoted as A). Let A_λ be a non-zero eigenspace of A corresponding to some eigenvalue $\lambda \neq 0$ (which must exist).

We define $\mathcal{M}$ to be the collection of subspaces of the form $M \cap A_\lambda$, where M is an $\mathcal{A}$-invariant subspace and $M \cap A_\lambda \neq \{0\}$. The class $\mathcal{M}$ is non-empty since $\overline{\mathcal{A}v} \cap A_\lambda \in \mathcal{M}$ for any $v \in A_\lambda$. Let M^* be an $\mathcal{A}$-invariant subspace such that $M^* \cap A_\lambda$ has the minimal dimension among all members of $\mathcal{M}$. Pick a non-zero vector $v \in M^* \cap A_\lambda$. We claim that the $\mathcal{A}$-invariant subspace $W \triangleq \overline{\mathcal{A}v}$ is irreducible.

In fact, suppose that E is a non-trivial $\mathcal{A}$-invariant subspace of W. Let E' be the orthogonal complement of E in W. Note that E' is also $\mathcal{A}$-invariant. One writes $v = v_1 + v_2$ where $v_1 \in E$ and $v_2 \in E'$. If $v_1 = 0$, then $v = v_2$ and thus $W \subseteq E' \subseteq W$, showing that $E = \{0\}$ which is a contradiction. Therefore, $v_1 \neq 0$. On the other hand, one has

$$\lambda v_1 + \lambda v_2 = \lambda v = Av = Av_1 + Av_2.$$

By the $\mathcal{A}$-invariance of E and E', one knows that $Av_1 \in E$, $Av_2 \in E^\perp$, and thus v_1, v_2 also belong to A_λ. If $v_2 \neq 0$, one knows that v is an element of $M^* \cap A_\lambda$ that does not belong to $E \cap A_\lambda$. In particular, one has

$$0 < \dim E \cap A_\lambda < \dim M^* \cap A_\lambda,$$

which contradicts the definition of M^*. Therefore, $v_2 = 0$. As a result, one has $v = v_1$ and thus $W \subseteq E \subseteq W$. This yields $E = W$ and concludes the irreducibility of W.

(ii) Suppose on the contrary that H_i occurs ($\mathcal{A}$-isomorphically) for infinitely many times in the decomposition (3.4), say

$$H_1 \cong H_2 \cong \cdots \cong H_n \cong \cdots \tag{3.5}$$

and all these components are orthogonal. As in Part (i), one can find $A \in \mathcal{A}$ such that $A|_{H_1} \neq 0$ and A is self-adjoint. Let $v_1 \in H_1$ be a unit eigenvector of A corresponding to some eigenvalue $\lambda \neq 0$. Define $v_n \in H_n$ $(n \geqslant 2)$ to be the image of v_1 under the isomorphism (3.5). Then v_n a unit λ-eigenvector of A for each n. It follows from the compactness of A that $\{Av_n : n \geqslant 1\}$ has a convergent subsequence. But this is not possible since

$$Av_n = \lambda v_n \in H_n$$

and the subspaces H_n's are orthogonal to each other. $\square$

3.3 The Peter-Weyl theorem

In this subsection, we establish the structure theorem for unitary representations of a compact Hausdorff group G (the Peter-Weyl theorem). This is the core of Fourier analysis on G. The main result is stated as follows, which contains several major parts.

Theorem 3.1 (The Peter-Weyl theorem)**.** *Let G be a compact Hausdorff group and let $T : G \to Aut(L^2(G))$ be its regular representation. Then the following statements hold true.*

(i) All unitary irreducible representations of G are finite dimensional.
(ii) The Hilbert space $L^2(G)$ is completely reducible for T, i.e. $L^2(G)$ can be written as an orthogonal direct sum of non-trivial T-irreducible subspaces:

$$L^2(G) = \bigoplus_{i \in \mathcal{I}} H_i. \tag{3.6}$$

In addition, each irreducible subspace H_i arising in the decomposition is finite dimensional and occurs for at most finitely many times (up to G-isomorphism).
(iii) Let $\sigma : G \to Aut(H)$ be a unitary irreducible representation and $H \neq \{0\}$. Then H occurs in the expansion (3.6) in the sense that there exists a component H_i in the decomposition (3.6) such that H is G-isomorphic to H_i.
(iv) Let $\sigma : G \to Aut(H)$ be a unitary representation. Then H is completely reducible for the representation σ.

There is a more quantitative part of the Peter-Weyl theorem telling us how to generate the space $L^2(G)$ by certain "special" functions arising from unitary irreducible representations, and these functions share some interesting orthogonality properties. This will be discussed in Section 3.3.3 below. The rest of this subsection is devoted to the proof of Theorem 3.1.

3.3.1 Finite dimensionality of irreducible representations

We first prove Part (i) of Theorem 3.1, which is a quite surprising fact on its own.

Theorem 3.2. *Let $\pi : G \to Aut\ (H)$ be a unitary irreducible representation. Then H is finite dimensional.*

Proof. Let u be a non-zero unit vector in H, and let $P : H \to H$ denote the projection operator onto the one dimensional subspace $\mathrm{Span}\{u\}$. Define a bounded linear operator $Q : H \to H$ by

$$Q(v) \triangleq \int_G \pi(x^{-1}) P \pi(x) v dx, \quad v \in H.$$

By using the fact that π is unitary and P is self-adjoint, it is easily checked that Q is a self-adjoint and commutes with $\pi(x)$ for all $x \in G$. Since π is irreducible, from Schur's lemma (cf. Theorem C.4 in Appendix C) one knows that $Q = c \cdot \mathrm{Id}$ for some $c \in \mathbb{C}$. Since

$$\langle Qu, u \rangle_H = \int_G \langle P\pi(x)u, \pi(x)u \rangle_H dx = \int_G \|P\pi(x)u\|_H^2 dx,$$

it is clear that $Q \neq 0$ and thus $c \neq 0$.

We assume on the contrary that H is infinite dimensional. Let $\{e_n : n \geqslant 1\}$ be an ONB of H. Given $x \in G$, let $e_n' \triangleq \pi(x)e_n$. Note that $\{e_n' : n \geqslant 1\}$ is also an ONB of H. It follows that

$$\langle \pi^{-1}(x) P \pi(x) e_n, e_n \rangle_H = \langle P e_n', e_n' \rangle_H = \langle u, e_n' \rangle_H^2,$$

and thus

$$\sum_{n=1}^{\infty} \langle \pi^{-1}(x) P \pi(x) e_n, e_n \rangle_H = \sum_{n=1}^{\infty} \langle u, e_n' \rangle_H^2 = \|u\|_H^2 = 1.$$

Therefore,

$$\begin{aligned} \infty = c \cdot \sum_{n=1}^{\infty} \langle e_n, e_n \rangle_H &= \sum_{n=1}^{\infty} \langle Q e_n, e_n \rangle_H \\ &= \int_G \sum_{n=1}^{\infty} \langle \pi^{-1}(x) P \pi(x) e_n, e_n \rangle_H dx = 1, \end{aligned}$$

which is absurd. Consequently, H must be finite dimensional. □

Remark 3.1. A careful examination of the proof shows that Q is a self-adjoint positive-definite operator that is of trace class. As a result, it is a compact operator. However, a compact operator cannot be a scalar multiple of the identity operator unless the underlying space is finite dimensional.

Due to the above finite dimensionality property, one has the following simple but quite useful fact for unitary irreducible representations.

Proposition 3.2. *(i) Suppose that $\sigma : G \to \mathrm{Aut}(H_1)$ and $\pi : G \to \mathrm{Aut}(H_2)$ are two unitary irreducible representations. Let $\Phi : H_1 \to H_2$ be a G-homomorphism. Then either $\Phi = 0$ or it is a G-isomorphism.*
(ii) Suppose that $\pi : G \to \mathrm{Aut}(H)$ is a unitary irreducible representation and $\Phi : H \to H$ is a G-homomorphism. Then $\Phi = c \cdot \mathrm{Id}$ for some $c \in \mathbb{C}$.

Proof. (i) The main observation is that $\mathrm{Ker}\Phi$ is an invariant subspace of H_1 and $\mathrm{Im}\Phi$ is an invariant subspace of H_2. If Φ is non-trivial, then $\mathrm{Ker}\Phi = \{0\}$ and $\mathrm{Im}\Phi = H_2$ since both of σ and π are irreducible. This implies that Φ is an isomorphism.
(ii) Let $c \in \mathbb{C}$ be an eigenvalue of Φ which exists since H is finite dimensional. Then $\Phi - c \cdot \mathrm{Id}$ is a G-homomorphism but not an isomorphism. According to Part (i), one has $\Phi - c \cdot \mathrm{Id} = 0$. □

Remark 3.2. Proposition 3.2 is also known as *Schur's lemma*. The proof is rather simple due to finite dimensionality. It is possible to prove the result without using finite dimensionality – in this case one needs to rely on the functional Schur's lemma in the context of Hilbert spaces (cf. Theorem C.4 in Appendix C).

The following property is the converse of Proposition 3.2, Part (ii). It provides a criterion for irreducibility which will be useful when we study the example of $G = \mathrm{SU}(2)$ (cf. Lemma 3.12 in Section 3.5.2 below).

Lemma 3.2. *Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary representation. Suppose that every G-homomorphism on H is a scalar multiple of the identity map. Then π is irreducible.*

Proof. Let W be a G-invariant subspace of H, and let $P : H \to H$ denote the orthogonal projection onto W. It is easy to see that $W^\perp$ is also a G-invariant subspace, and as a result P is a G-homomorphism. By the assumption, one knows that $P = c \cdot \mathrm{Id}$ for some $c \in \mathbb{C}$. This clearly implies that $W = \{0\}$ (if $c = 0$) or $W = H$ (if $c \neq 0$). □

3.3.2 Induced representations of $\mathcal{C}(G)$ and complete reducibility of the regular representation

The proof of the Peter-Weyl theorem uses a crucial idea of induced representations of continuous functions which we now describe. Let G be a compact Hausdorff group.

Let $\mathcal{C}(G)$ denote the space of complex-valued continuous functions on G. Given $f, g \in \mathcal{C}(G)$, their *convolution* is defined as

$$\varphi * \psi(x) \triangleq \int_G \varphi(xy^{-1})\psi(y)dy.$$

It follows that $(\mathcal{C}(G), *)$ is an algebra:

$$\varphi, \psi \in \mathcal{C}(G), c \in \mathbb{C} \implies c\varphi + \psi, \varphi * \psi \in \mathcal{C}(G).$$

We also set $\varphi^*(x) \triangleq \overline{\varphi(x^{-1})}$ for $\varphi \in \mathcal{C}(G)$.

Suppose that $\pi : G \to \mathrm{Aut}(H)$ be a representation of G on some Banach space H. Given $\varphi \in \mathcal{C}(G)$, the representation π induces a natural action of φ on the space H as endomorphisms (continuous linear transformations):

$$\pi^1(\varphi)v = \int_G \varphi(x)\pi(x)v dx, \quad v \in H.$$

Proposition 3.3. *The action* $\pi^1 : (\mathcal{C}(G), *) \to (\mathrm{End}(H), \circ)$ *is an algebra homomorphism. In addition, if* π *is a unitary representation of* G *on a Hilbert space* H*, then*

$$\pi^1(\varphi)^* = \pi^1(\varphi^*) \tag{3.7}$$

for all $\varphi \in \mathcal{C}(G)$.

Proof. We first check that $\pi^1(\varphi)$ is a bounded linear operator on H. Indeed, from Theorem C.1 one knows that

$$M \triangleq \sup_{x \in G} \|\pi(x)\|_{H \to H} < \infty.$$

Therefore,

$$\|\pi^1(\varphi)v\|_H \leqslant M\|\varphi\|_\infty \|v\|_H \quad \forall v \in H.$$

The linearity of π^1 is obvious. We now show that π^1 is an algebra homomorphism. To see this, by definition one has

$$
\begin{aligned}
\pi^1(\varphi * \psi)(v) &= \int_G \varphi * \psi(z)\pi(z)v dz \\
&= \int_G \big(\int_G \varphi(zy^{-1})\psi(y)dy\big)\pi(z)vdz \\
&= \int_G \psi(y)dy \int_G \varphi(zy^{-1})\pi(z)vdz \\
&= \int_G \psi(y)dy \int_G \varphi(x)\pi(xy)vdx \quad (z = xy) \\
&= \int_G \varphi(x)\pi(x)\big(\int_G \psi(y)\pi(y)vdy\big)dx \\
&= \big(\pi^1(\varphi) \circ \pi^1(\psi)\big)(v).
\end{aligned}
$$

Finally, we prove the relation (3.7). Let $v, w \in H$ and $\varphi \in \mathcal{C}(G)$. Then

$$
\begin{aligned}
\langle \pi^1(\varphi)v, w\rangle_H &= \int_G \varphi(x)\langle \pi(x)v, w\rangle_H dx \\
&= \int_G \varphi(x)\langle v, \pi(x^{-1})w\rangle_H dx \\
&= \int_G \overline{\varphi^*(x)}\langle v, \pi(x)w\rangle_H dx \quad (x \mapsto x^{-1}) \\
&= \int_G \langle v, \varphi^*(x)\pi(x)w\rangle_H dz \\
&= \langle v, \pi^1(\varphi^*)w\rangle_H.
\end{aligned}
$$

The relation (3.7) thus follows. □

Remark 3.3. Although we mostly work with $\mathcal{C}(G)$-actions, the same action by $L^1(G)$-functions is also well defined since G is compact.

Equivalence between G- and $\mathcal{C}(G)$-actions

The actions of G and $\mathcal{C}(G)$ are essentially the same thing, as one can recover the former from the latter by using a standard idea of approximation of identity.

Definition 3.8. Let $x \in G$. A *Dirac sequence* with respect to x is a sequence of functions $\{\varphi_n : n \geqslant 1\} \subseteq \mathcal{C}(G)$ such that:

(i) $\varphi_n \geqslant 0$;
(ii) $\int_G \varphi_n(x)dx = 1$;
(iii) for any neighbourhood V of x, one has $\mathrm{supp}\varphi_n \subseteq V$ when n is large enough.

Dirac sequences can be easily constructed by using bump functions. In addition, if φ_n is a Dirac sequence with respect to the identity e, then for each $x \in G$, the sequence $l_x\varphi_n(\cdot) \triangleq \varphi_n(x^{-1}\cdot)$ is a Dirac sequence with respect to x.

Using the notion of Dirac sequences, one can recover representations of G from the induced representations of $\mathcal{C}(G)$.

Proposition 3.4. *Let $\pi : G \to Aut(H)$ be a representation of G on some Banach space H. Let $x \in G$ and let $\{\varphi_n\}$ be a Dirac sequence with respect to x. Then for each $v \in H$, one has*

$$\lim_{n\to\infty} \pi^1(\varphi_n)v = \pi(x)v.$$

Proof. Given $\varepsilon > 0$, there exists a neighbourhood V of x such that

$$\|\pi(y)v - \pi(x)v\|_H < \varepsilon \quad \forall y \in V.$$

It follows that

$$\begin{aligned}
&\left\|\pi^1(\varphi_n)v - \pi(x)v\right\|_H \\
&= \Big\| \int_G \varphi_n(y)\pi(y)vdy - \int_G \varphi_n(y)\pi(x)vdy\Big\|_H \\
&= \Big\| \int_V \varphi_n(y)\big(\pi(y)v - \pi(x)v\big)dy\Big\|_H \\
&\leqslant \varepsilon \cdot \int_V \varphi_n(y)dy \\
&\leqslant \varepsilon,
\end{aligned}$$

provided that n is large enough. $\square$

A direct corollary of Proposition 3.4 is the following result.

Corollary 3.1. *Let $\pi : G \to \mathrm{Aut}(H)$ be a representation of G on some Banach space H and let W be a closed subspace of H. Then W is G-invariant if and only if it is $\mathcal{C}(G)$-invariant. In particular, W is G-irreducible if and only if it is $\mathcal{C}(G)$-irreducible.*

Remark 3.4. Representations of continuous functions extend to the case when G is locally compact. In this case, one needs to replace the algebra $\mathcal{C}(G)$ by $\mathcal{C}_c(G)$ (compactly supported continuous functions).

$\mathcal{C}(G)$-action on $L^2(G)$ as compact operators

We now consider the action of $\mathcal{C}(G)$ induced by the regular representation $T : G \to \mathrm{Aut}(L^2(G))$. Given a function $\varphi \in \mathcal{C}(G)$, we denote $\varphi^-(x) \triangleq \varphi(x^{-1})$.

Lemma 3.3. *The induced representation T^1 of $\mathcal{C}(G)$ on $L^2(G)$ is given by*

$$T^1(\varphi)f = f * \varphi^-, \quad \forall \varphi \in \mathcal{C}(G), f \in L^2(G).$$

Proof. By definition, one has

$$\begin{aligned}(T^1(\varphi)f)(x) &= \int_G \varphi(y)(T(y)f)(x)dy = \int_G \varphi(y)f(xy)dy \\ &= \int_G f(xy^{-1})\varphi(y^{-1})dy \\ &= f * \varphi^-(x).\end{aligned} \tag{3.8}$$

□

A major benefit of considering the induced $\mathcal{C}(G)$-action is that it gives rise to compact operators in the case of the regular representation.

Lemma 3.4. *For each $\varphi \in \mathcal{C}(G)$, the induced action $T^1(\varphi) : L^2(G) \to L^2(G)$ is a compact operator.*

Proof. The main idea is to see that $T^1(\varphi)$ can be approximated by operators with finite rank (i.e. having finite dimensional range). By a change of variables, one first rewrites (3.8) as

$$(T^1(\varphi)f)(x) = \int_G f(y)\varphi(x^{-1}y)dy. \tag{3.9}$$

In particular, the action of φ on $L^2(G)$ is given by a kernel $(x, y) \mapsto \varphi(x^{-1}y)$.

Let $\mathcal{C}(G \times G)$ denote the space of continuous functions on $G \times G$. Consider the family $\mathcal{A} \subseteq C(G \times G)$ of functions defined by

$$\mathcal{A} = \mathrm{Span}\big\{\varphi(x)\psi(y) : \varphi, \psi \in \mathcal{C}(G)\big\}.$$

It follows that $\mathcal{A}$ is an algebra (under the pointwise multiplication) and it separates points in $G \times G$. According to the Stone-Weierstrass theorem (cf. Theorem C.2

in Appendix C), $\mathcal{A}$ is dense in $\mathcal{C}(G\times G)$ under the uniform topology. Note that each $\Phi\in\mathcal{C}(G\times G)$ induces an integral operator on $L^2(G)$ defined by

$$T_\Phi(f)(x)\triangleq\int_G f(y)\Phi(x,y)dy.$$

It is easy to see that

$$\Phi_n\to\Phi \text{ uniformly} \implies T_{\Phi_n}\to T_\Phi \text{ as bounded linear operators on } L^2(G).$$

In particular, the induced action $T^1(\varphi)$ given by (3.9) can be approximated by the operators T_Φ (under the operator norm) where $\Phi\in\mathcal{A}$.

On the other hand, if $\Phi(x,y)\triangleq\varphi(x)\psi(y)$, the integral operator T_Φ is given by

$$T_\Phi(f)(x)=\int_G f(y)\varphi(x)\psi(y)dy=\big(\int_G f(y)\psi(y)dy\big)\cdot\varphi(x).$$

In particular, T_Φ has a one dimensional range. It follows that the integral operators on $L^2(G)$ induced by the elements in $\mathcal{A}$ are all of finite rank. As a result, one sees that $T^1(\varphi)$ can be approximated by operators of finite rank, and is thus a compact operator. □

Complete reducibility of the regular representation

Proposition 3.1 and Corollary 3.1 naturally lead us to the complete reducibility of the regular representation, yielding the proof of Part (ii) of the Peter-Weyl theorem.

Theorem 3.3. *The regular representation $T:G\to Aut(L^2(G))$ is completely reducible, i.e. $L^2(G)$ can be written as an orthogonal direct sum of non-trivial T-irreducible subspaces. In addition, each irreducible subspace arising in the decomposition is finite dimensional and occurs for at most finitely many times (up to G-isomorphism).*

Proof. From Proposition 3.3 and Lemma 3.4, one sees that $T^1(\mathcal{C}(G))$ is an algebra of compact operators acting on $L^2(G)$. In addition, since T is unitary, the relation (3.7) shows that $T^1(\mathcal{C}(G))$ is $*$-closed. It follows from Proposition 3.1 that $L^2(G)$ admits an orthogonal decomposition into non-trivial $\mathcal{C}(G)$-irreducible subspaces. Let H_π be any component appearing in the decomposition . From Corollary 3.1, one knows that $\mathcal{C}(G)$-irreducibility and G-irreducibility are equivalent. According to Theorem 3.2, H_π is finite dimensional. Finally, since $\pi(e)=\mathrm{Id}$, using approximation of identity one sees that at least one member of $\mathcal{C}(G)$ acts non-trivially on H_π. Therefore, H_π occurs for at most finitely many times (up to G-isomorphism). □

3.3.3 Matrix coefficients and their denseness in $L^2(G)$

The Peter-Weyl theorem has a more quantitative part involving the so-called matrix coefficients which we now describe.

Definition 3.9. Let $\pi : G \to \mathrm{Aut}(H)$ be a representation of G on some Banach space H. Given $\lambda \in H^*$ and $v \in H$, the continuous function

$$\pi_{\lambda,v}(x) \triangleq \lambda(\pi(x)v), \quad x \in G$$

is called a *coefficient function* associated with π. When H is a Hilbert space, λ is represented by a vector w and one can also write

$$\pi_{\lambda,v}(x) = \langle \pi(x)v, w\rangle_H.$$

A *matrix coefficient* of G is a coefficient function associated with a finite dimensional representation of G.

The aim of this part is to prove the following result, which is part of the Peter-Weyl theorem that is not stated in Theorem 3.1.

Theorem 3.4. *Matrix coefficients of G are dense in $L^2(G)$.*

To prove the theorem, we first introduce the following useful characterisation of matrix coefficients.

Lemma 3.5. *Let $f \in \mathcal{C}(G)$. Then f is a matrix coefficient of G if and only if $\{T(y)f : y \in G\}$ span a finite dimensional vector space.*

Proof. Necessity. Suppose that f is a matrix coefficient of G, say $f(x) = \lambda(\pi(x)v)$ for some finite dimensional representation $\pi : G \to \mathrm{Aut}(H)$. Then for each given $y \in G$, one has

$$(T(y)f)(x) = f(xy) = \lambda(\pi(x)\pi(y)v), \quad x \in G.$$

In particular, $T(y)f$ is also a matrix coefficient. But since $\dim H < \infty$, the space of all coefficient functions associated with π has dimension at most $(\dim H)^2$. Therefore, $\{T(y)f : y \in G\}$ span a finite dimensional space.

Sufficiency. Let $H \triangleq \mathrm{Span}\{T(y)f : y \in G\} \subseteq \mathcal{C}(G)$. Note that the restriction of T on H is a finite dimensional representation of G. Define $\lambda \in H^*$ by $\lambda(\varphi) \triangleq \varphi(e)$. Then

$$\lambda(T(x)f) = (T(x)f)(e) = f(x), \quad x \in G.$$

As a result, one sees that f is a matrix coefficient. $\square$

In addition to Lemma 3.5, our proof of Theorem 3.4 also relies on the following nice observation which provides a natural way of constructing (finite dimensional) T-invariant subspaces of $L^2(G)$. Consider the following counterpart of the regular representation (defined by left translation instead):

$$L : L^2(G) \to L^2(G),\ (L(y)f)(x) \triangleq f(y^{-1}x).$$

Given $\varphi \in \mathcal{C}(G)$, recall that $L^1(\varphi)$ is the induced representation of φ. Similar to Lemma 3.3, it can be shown that $L^1(\varphi)f = \varphi * f$. In particular, $L^1(\varphi)$ is a compact operator on $L^2(G)$. Suppose further that φ is symmetric, i.e. $\varphi(x) = \varphi(x^{-1})$. Then $L^1(\varphi)$ is self-adjoint. According to the spectral theorem (cf. Theorem C.3 in Appendix C), one can write

$$L^2(G) = \bigoplus_{\lambda} H_\lambda, \tag{3.10}$$

where H_λ are orthogonal λ-eigenspaces of $L^1(\varphi)$ and $\dim H_\lambda < \infty$ when $\lambda \neq 0$. Here the set $\{\lambda : H_\lambda \neq \{0\}\}$ is a countable subset of $\mathbb{R}$ with $\lambda = 0$ being the only possible accumulation point.

Lemma 3.6. *For each $\lambda \neq 0$, the subspace H_λ is T-invariant.*

Proof. Let $f \in H_\lambda$ so that

$$(L^1(\varphi)f)(u) = \int_G \varphi(uv^{-1})f(v)dv = \lambda f(u), \quad u \in G.$$

Then

$$\begin{aligned}(L^1(\varphi)T(x)f)(y) &= \int_G \varphi(yz^{-1})(T(x)f)(z)dz \\ &= \int_G \varphi(yz^{-1})f(zx)dz \\ &= \int_G \varphi(yxv^{-1})f(v)dv \\ &= \lambda f(yx) = \lambda(T(x)f)(y).\end{aligned}$$

Therefore, $T(x)f \in H_\lambda$. □

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4. We prove a stronger result that the matrix coefficients are dense in $\mathcal{C}(G)$ under the uniform topology. The L^2-denseness follows from this trivially. Let $f \in \mathcal{C}(G)$ be fixed. Given $\varepsilon > 0$, by applying Proposition 3.4 to the action L on the Banach space $(\mathcal{C}(G), \|\cdot\|_\infty)$, one can find $\varphi \in \mathcal{C}(G)$ such that

$$\|L^1(\varphi)f - f\|_\infty < \varepsilon.$$

Note that φ is constructed from a Dirac sequence with respect to the identity e, and is thus supported in a small neighbourhood of e. By considering $\frac{\varphi(x)+\varphi(x^{-1})}{2}$ if necessary, one may assume that $\varphi(x) = \varphi(x^{-1})$. In this case, $L^1(\varphi)$ is compact and self-adjoint. Let (3.10) be its spectral decomposition. For each $\eta > 0$, we define

$$W_\eta \triangleq \bigoplus_{\lambda:|\lambda|>\eta} H_\lambda.$$

According to Lemma 3.6, W_η is a finite dimensional, T-invariant subspace of $L^2(G)$. It follows from Lemma 3.5 that elements in W_η are matrix coefficients of G.

Let f_1 denote the projection of f onto $H_0 = \ker L^1(\varphi)$. To complete the proof, one chooses η small enough so that

$$\|f - f_1 - f_2\|_{L^2} < \frac{\varepsilon}{\|\varphi\|_\infty},$$

where f_2 is the projection of f onto the subspace W_η. It follows that $L^1(\varphi)(f_2) \in W_\eta$ is a matrix coefficient of G. In addition, one has

$$\begin{aligned}
\|f - L^1(\varphi)(f_2)\|_\infty &\leqslant \|f - L^1(\varphi)(f)\|_\infty + \|L^1(\varphi)(f - f_2)\|_\infty \\
&= \|f - L^1(\varphi)(f)\|_\infty + \|L^1(\varphi)(f - f_1 - f_2)\|_\infty \\
&\leqslant \varepsilon + \|\varphi\|_\infty \cdot \|f - f_1 - f_2\|_{L^1} \\
&\leqslant \varepsilon + \|\varphi\|_\infty \cdot \|f - f_1 - f_2\|_{L^2} \\
&< 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, one co that f can be approximated uniformly by matrix coefficients on G. This completes the proof of the theorem.

□

3.3.4 Schur's orthogonality relations

Our next step is to understand several orthogonal relations for the matrix coefficients as well as the so-called characters. Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary irreducible representation. Recall from Theorem 3.2 that $\dim H < \infty$. Let

$\{e_1, \cdots, e_n\}$ be an ONB of H. The functions

$$\{\pi_{ij}(x) \triangleq \langle \pi(x)e_i, e_j \rangle : 1 \leqslant i, j \leqslant n\}$$

form a natural family of matrix coefficients associated with the representation π (as we will see, they are orthogonal to each other).

Definition 3.10. The *character* of π is defined by its trace function:

$$\chi(x) \triangleq \mathrm{Tr}(\pi(x)) = \sum_{i=1}^{n} \pi_{ii}(x), \quad x \in G.$$

Note that G-isomorphic representations have the same character.

We would like to know how the coefficient functions and the character (as functions in $\mathcal{C}(G)$) of a given representation act on another arbitrary representation. The key observation is contained in the lemma below. Recall that $\varphi^{-}(x) \triangleq \varphi(x^{-1})$ for $\varphi \in \mathcal{C}(G)$.

Lemma 3.7. *Let $\sigma : G \to Aut(H_1)$ and $\pi : G \to Aut(H_2)$ be two representations of G on Banach spaces H_1, H_2 respectively. Let $\lambda \in H_1^*$ and $w \in H_2$ be fixed. Then the operator*

$$v \mapsto L_{\lambda,w}(v) \triangleq \pi^1(\sigma_{\lambda,v}^{-})(w)$$

defines a G-homomorphism from H_1 to H_2.

Proof. This is simple unwinding of definition: for each $x \in G$ one has

$$\begin{aligned}
\pi(x)L_{\lambda,w}(v) &= \int_G \sigma_{\lambda,v}(y^{-1})\pi(xy)w dy \\
&= \int_G \lambda\big(\sigma(y^{-1})v\big)\pi(xy)w dy \\
&= \int_G \lambda\big(\sigma(z^{-1})\sigma(x)v\big)\pi(z)w dz \\
&= L_{\lambda,w}(\sigma(x)v).
\end{aligned}$$

□

Lemma 3.7 together with Schur's Lemma (cf. Proposition 3.2) allow us to describe actions of coefficient functions and characters easily. Let $\sigma : G \to \mathrm{Aut}(H_1)$ and $\pi : G \to \mathrm{Aut}(H_2)$ be two unitary irreducible representations. In what follows, we discuss the cases when σ, π are *distinct* (i.e. non-G-isomorphic) and G-isomorphic separately. Note that we are now in the context of matrix coefficients since H_1, H_2 are finite dimensional.

The non-isomorphic case

Suppose that $\sigma : G \to \mathrm{Aut}(H_1)$ and $\pi : G \to \mathrm{Aut}(H_2)$ are distinct, unitary irreducible representations. In this case, one has the following orthogonality property for the matrix coefficients.

Proposition 3.5. *Let $\lambda \in H_1^*$ and $v \in H_1$. Then the function $\sigma_{\lambda,v}^-$ acts trivially on H_2. In addition, matrix coefficients associated with σ and π are orthogonal to each other:*

$$\langle \sigma_{\lambda,v}, \pi_{\mu,w} \rangle_{L^2} = \int_G \sigma_{\lambda,v}(x)\overline{\pi_{\mu,w}(x)}dx = 0 \tag{3.11}$$

for any $\lambda \in H_1^*, v \in H_1$ and $\mu \in H_2^*, w \in H_2$.

Proof. From Lemma 3.7, one knows that

$$L_{\lambda,w}(v) \triangleq \int_G \sigma_{\lambda,v}^-(x)\pi(x)wdx$$

is a G-homomorphism. Since H_1 and H_2 are not G-isomorphic, from Proposition 3.2 one co that $L_{\lambda,w} = 0$. The first assertion follows by regarding v as fixed and $w \in H_2$ as a variable.

To prove the relation (3.11), let $\lambda \in H_1^*, v \in H_1$ and $\mu \in H_2^*, w \in H_2$ be given fixed. Suppose that $w' \in H_2$ is the vector representing μ, i.e. $\mu(\cdot) = \langle \cdot, w_2 \rangle_{H_2}$. Then one has

$$\begin{aligned}
\langle \sigma_{\lambda,v}, \pi_{\mu,w} \rangle_{L^2} &= \int_G \lambda(\sigma(x)v)\overline{\langle \pi(x)w, w' \rangle_{H_2}}dx \\
&= \int_G \lambda(\sigma(x)v)\overline{\langle w, \pi(x^{-1})w' \rangle_{H_2}}dx \quad (\pi \text{ is unitary}) \\
&= \int_G \lambda(\sigma(x)v)\langle \pi(x^{-1})w', w \rangle_{H_2}dx \\
&= \int_G \lambda(\sigma(x^{-1})v)\langle \pi(x)w', w \rangle_{H_2}dx \\
&= 0,
\end{aligned}$$

where the last step follows from the fact that

$$\int_G \lambda(\sigma(x^{-1})v)\pi(x)w'dx = L_{\lambda,w'}(v) = 0.$$

Therefore, the relation (3.11) follows. □

Next, we consider the action of characters. Let χ_σ and χ_π be the characters of σ and π respectively.

Proposition 3.6. *The function χ_σ^- acts trivially on H_2. In addition, one has the following orthogonality relations:*

$$\chi_\sigma * \chi_\pi = 0, \ \langle \chi_\sigma, \chi_\pi \rangle_{L^2} = 0. \tag{3.12}$$

Proof. Since the character is a sum of matrix coefficients, the first assertion follows directly from Proposition 3.5. For the second assertion, note from Proposition 3.5 that

$$\int_G \lambda(\sigma(ax^{-1})v)\pi(x)w dx = 0$$

for any $a \in G$ (consider $v \mapsto \lambda(\sigma(a)v)$ as the functional λ in the proposition). Therefore,

$$\int_G \lambda(\sigma(ax^{-1})v)\mu(\pi(x)w)dx = 0 \tag{3.13}$$

for any λ, v, a, μ, w belonging to the relevant spaces. Now let $\{e_1, \cdots, e_m\}$ and $\{e_1', \cdots, e_n'\}$ be ONB's of H_1 and H_2 respectively. It follows from (3.13) that

$$\begin{aligned} \chi_\sigma * \chi_\pi(a) &= \int_G \chi_\sigma(ax^{-1})\chi_\pi(x)dx \\ &= \sum_{i,j} \int_G \langle \sigma(ax^{-1})e_i, e_i\rangle_{H_1} \langle \pi(x)e_j', e_j'\rangle_{H_2} dx \\ &= 0. \end{aligned} \tag{3.14}$$

This yields the first relation in (3.12). The second relation follows from (3.14) by taking $a = e$, together with the fact that

$$\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}. \tag{3.15}$$

$\square$

Corollary 3.2. *Let $\pi : G \to \mathrm{Aut}(H)$ be a non-trivial, unitary irreducible representation. Then*

$$\int_G \lambda(\pi(x)v)dx = 0 \quad \forall \lambda \in H^*, v \in H.$$

In particular,

$$\int_G \chi(x)dx = 0$$

where χ is the character of π.

Proof. The trivial unitary representation

$$\sigma : G \to S^1,\ \sigma(x) = 1\ \forall x$$

has matrix coefficients

$$\sigma_{\lambda,v}(x) = \lambda(v), \quad x \in G$$

and character $\chi_\sigma = 1$. Since π is non-trivial, it is not G-isomorphic to the trivial representation. The result follows from Proposition 3.5 and Proposition 3.6. □

The isomorphic case

We now examine the situation when the two representations σ, π are G-isomorphic. Without loss of generality, one may assume that $\sigma = \pi$ and $H_1 = H_2 = H$. We are interested in how the matrix coefficients and character act on H. Let $d_\pi \triangleq \dim H$.

We first consider the action of matrix coefficients.

Proposition 3.7. *For any $\lambda \in H^*$, $v \in H$, the action of $\pi^-_{\lambda,v}$ on H is the rank-one projection operator onto* $\mathrm{Span}\{v\}$ *given by*

$$\pi^1(\pi^-_{\lambda,v})(w) = \frac{\lambda(w)}{d_\pi} v, \quad w \in H.$$

Proof. Given $\lambda \in H^*$ and $w \in H$, recall from Lemma 3.7 that the operator

$$v \mapsto L_{\lambda,w}(v) \triangleq \pi^1(\pi^-_{\lambda,v})(w)$$

defines a G-homomorphism on H. According to Proposition 3.2 (ii), one has

$$L_{\lambda,w}(v) = cv, \quad v \in H \tag{3.16}$$

for some constant $c \in \mathbb{C}$.

To complete the proof, it remains to determine the constant c. For this purpose, we take trace on both sides of (3.16). Note that

$$\mathrm{Tr}(L_{\lambda,w}) = \int_G \mathrm{Tr}[v \mapsto \lambda(\pi(x^{-1})v)\pi(x)w]dx.$$

The operator inside the trace function on the right hand side is of the form

$$v \mapsto \mu(v)w'$$

where μ is a linear functional on H and $w' \in H$ is fixed. Simple linear algebra shows that the trace of such an operator is given by $\mu(w')$. Therefore,

$$\mathrm{Tr}(L_{\lambda,w}) = \int_G \lambda(\pi(x^{-1})\pi(x)w)dx = \lambda(w) \cdot \int_G dx = \lambda(w).$$

It follows from (3.16) that $\lambda(w) = c \cdot d_\pi$, or equivalently, $c = \frac{\lambda(w)}{d_\pi}$. □

Proposition 3.7 implies the following important orthogonality relation for matrix coefficients. Let $\{e_1, \cdots, e_{d_\pi}\}$ be a given ONB of H.

Proposition 3.8. *The associated matrix coefficients*

$$\pi_{ij}(x) \triangleq \langle \pi(x)e_i, e_j \rangle_H, \quad 1 \leqslant i, j \leqslant d_\pi$$

are orthogonal to each other. More precisely, one has

$$\langle \pi_{ij}, \pi_{kl} \rangle_{L^2} = \frac{1}{d_\pi}\delta_{ik}\delta_{jl}. \tag{3.17}$$

Proof. According to Proposition 3.7, one has

$$\begin{aligned}
\langle \pi_{ij}, \pi_{kl} \rangle_{L^2} &= \int_G \langle \pi(x)e_i, e_j \rangle_H \cdot \overline{\langle \pi(x)e_k, e_l \rangle_H} dx \\
&= \int_G \langle \pi(x^{-1})e_l, e_k \rangle_H \langle \pi(x)e_i, e_j \rangle_H dx \\
&= \frac{1}{d_\pi} \langle e_i, e_k \rangle_H \langle e_l, e_j \rangle_H \\
&= \frac{1}{d_\pi}\delta_{ik}\delta_{jl}.
\end{aligned}$$

□

Proposition 3.7 also immediately yields the action of the character.

Proposition 3.9. *Let χ_π be the character of π. Then*

$$\pi^1(\chi_\pi^-) = \frac{1}{d_\pi}\mathrm{Id}.$$

In addition, one has

$$\chi_\pi * \chi_\pi = \frac{1}{d_\pi}\chi_\pi, \ \ \langle \chi_\pi, \chi_\pi \rangle_{L^2} = 1. \tag{3.18}$$

Proof. Let $\{e_1, \cdots, e_{d_\pi}\}$ be an ONB of H. From Proposition 3.7, one has

$$\pi^1(\chi_\pi^-)(v) = \sum_{i=1}^{d_\pi} \pi^1(\pi_{e_i,e_i}^-) = \sum_{i=1}^{d_\pi} \frac{\langle v, e_i\rangle_H}{d_\pi} e_i = \frac{v}{d_\pi}.$$

The first assertion thus follows. The argument for the second assertion follows the same line as the proof of (3.12). According to Proposition 3.7, one has

$$\int_G \lambda(\pi(a)\pi(x^{-1})v)\pi(x)w dx = \frac{\lambda(\pi(a)w)}{d_\pi} v$$

for any $\lambda \in H^*$, $v, w \in H$ and $a \in G$. Given $a \in G$, note that

$$\{e_i' \triangleq \pi(a^{-1})e_i : 1 \leqslant i \leqslant d_\pi\}$$

is also an ONB of H. Therefore,

$$\begin{aligned}
\chi_\pi * \chi_\pi(a) &= \int_G \chi_\pi(ax^{-1})\chi_\pi(x)dx \\
&= \sum_{i,j=1}^{d_\pi} \int_G \langle \pi(ax^{-1})e_i, e_i\rangle_H \langle \pi(x)e_j', e_j'\rangle_H dx \\
&= \sum_{i,j=1}^{d_\pi} \frac{1}{d_\pi} \langle e_j', \pi(a^{-1})e_i\rangle_H \langle e_i, e_j'\rangle_H \\
&= \sum_{i=1}^{d_\pi} \frac{1}{d_\pi} \langle e_i, \pi(a^{-1})e_i\rangle_H = \sum_{i=1}^{d_\pi} \frac{1}{d_\pi} \langle \pi(a)e_i, e_i\rangle_H \\
&= \frac{1}{d_\pi}\chi_\pi(a).
\end{aligned}$$

The first part of (3.18) thus follows. The second part is obtained by taking $a = e$, together with the observations (3.15) and

$$\chi_\pi(e) = \sum_{i=1}^{d_\pi} \langle \pi(e)e_i, e_i\rangle_H = d_\pi.$$

$\square$

Summary of Schur's orthogonality relations

We summarise the orthogonality relations for the matrix coefficients and characters as below. These results are often known as *Schur's orthogonality relations*. Let $\sigma : G \to \mathrm{Aut}(H_1)$ and $\pi : G \to \mathrm{Aut}(H_2)$ be two unitary irreducible representations. We write $\sigma \sim \pi$ (respectively, $\sigma \nsim \pi$) to denote the case when H_1 and H_2 are G-isomorphic (respectively, not G-isomorphic).

(i) Let $\lambda \in H_1^*, v \in H_1$. Then

$$\pi^1(\sigma^-_{\lambda,v})(\cdot) = \begin{cases} 0, & \sigma \nsim \pi; \\ \frac{\lambda(\Phi^{-1}(\cdot))\Phi(v)}{d_\pi}, & \sigma \sim \pi, \end{cases} \tag{3.19}$$

where $\Phi : H_1 \to H_2$ denotes the underlying G-isomorphism in the second case.
(ii) The characters χ_σ and χ_π satisfy the relation:

$$\pi^1(\chi^-_\sigma) = \begin{cases} 0, & \sigma \nsim \pi; \\ \frac{1}{d_\pi}\mathrm{Id}, & \sigma \sim \pi, \end{cases}$$

and

$$\chi_\sigma * \chi_\pi = \begin{cases} 0, & \sigma \nsim \pi; \\ \frac{1}{d_\pi}\chi_\pi, & \sigma \sim \pi, \end{cases} \quad \langle \chi_\sigma, \chi_\pi \rangle_{L^2} = \begin{cases} 0, & \sigma \nsim \pi; \\ 1, & \sigma \sim \pi. \end{cases} \tag{3.20}$$

3.3.5 Completing the proof of the Peter-Weyl theorem

Using the idea of matrix coefficients, we can now prove Part (iii) and Part (iv) of the Peter-Weyl theorem concerning with general representations.

Proof of Theorem 3.1, Part (iii). Let $\sigma : G \to \mathrm{Aut}(H)$ be a unitary irreducible representation and $H \neq \{0\}$. Suppose on the contrary that H does not occur in the decomposition (3.6) of $L^2(G)$. Let ψ be an arbitrary matrix coefficient associated with σ. According to Proposition 3.5, ψ acts trivially on each component of the $L^2(G)$-decomposition. As a result, ψ^- acts trivially on $L^2(G)$. In view of Lemma 3.3, one has

$$T^1(\psi^-)f = f * \psi = 0 \quad \forall f \in L^2(G).$$

A standard application of approximation of identity shows that $\psi = 0$. Since ψ is arbitrary, one finds that $H = \{0\}$ which is a contradiction. Therefore, H must occur in the decomposition (3.6).

□

Finally, we prove the complete irreducibility of a general unitary representation.

Proof of Theorem 3.1, Part (iv). Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary representation. In the same way as the proof of Proposition 3.1, the key point is to establish the existence of a non-trivial finite dimensional irreducible subspace of H. After that, the rest of the argument is a standard application of Zorn's lemma. Because of finite dimensionality, one only needs to prove the existence of a non-trivial finite dimensional π-invariant subspace.

To this end, let w be a non-zero vector in H. By using approximation of identity and the denseness of matrix coefficients (cf. Theorem 3.4), one can find a matrix coefficient f of G such that

$$\int_G f(x^{-1})\pi(x)w dx \neq 0.$$

Since f is a matrix coefficient, there is a finite dimensional representation $\sigma : G \to \mathrm{Aut}(H_1)$ and some $\lambda \in H_1^*, v_0 \in H_1$, such that

$$f(x) = \lambda(\sigma(x)v_0).$$

Recall from Lemma 3.7 that the linear operator $L_{\lambda,w} : H_1 \to H$ defined by

$$L_{\lambda,w}(v) \triangleq \int_G \lambda(\sigma(x^{-1})v)\pi(x)w dx, \quad v \in H_1$$

is a G-homomorphism. Since

$$L_{\lambda,w}(v_0) = \int_G f(x^{-1})\pi(x)w dx \neq 0,$$

one sees that $W \triangleq L_{\lambda,w}(H_1)$ is a non-trivial finite dimensional subspace of H, which is clearly π-invariant.

$\square$

3.4 Fourier analysis on compact Hausdorff groups

Having all the previous structures at hand, we can now discuss the associated Fourier analysis. Let G be a compact Hausdorff group. Let Π be the enumeration of isomorphism classes of unitary irreducible representations of G. Such an enumeration exists since all unitary irreducible representations are contained in the $L^2(G)$-decomposition according to the Peter-Weyl theorem. We sometimes write (π, H_π) to keep track of the underlying Hilbert space on which π acts.

3.4.1 The spectral decomposition

For each isomorphism class $[\pi] \in \Pi$, let $L^2(G)_\pi$ be the space of matrix coefficients associated with π. Note that $L^2(G)_\pi$ is a finite dimensional subspace (of dimension at most d_π^2) of $L^2(G)$ which depends only on the isomorphism class $[\pi]$.

Theorem 3.5. *(i) Given $[(\pi, H_\pi)] \in \Pi$, let $\{e_1, \cdots, e_{d_\pi}\}$ be an ONB of H_π. Then*

$$\{d_\pi^{1/2}\langle \pi(x)e_i, e_j\rangle_H : 1 \leqslant i, j \leqslant d_\pi\} \tag{3.21}$$

form an ONB of $L^2(G)_\pi$. In particular, one has

$$\dim L^2(G)_\pi = d_\pi^2.$$

(ii) The space $L^2(G)$ admits the following canonical decomposition:

$$L^2(G) = \bigoplus_{[\pi]\in\Pi} L^2(G)_\pi. \tag{3.22}$$

As a result of Part (i), $L^2(G)$ has an ONB given by orthonormal matrix coefficients associated with isomorphic classes of representations in Π.

Proof. Schur's orthogonality relations imply that (3.21) is an ONB of $L^2(G)_\pi$, and $L^2(G)_\sigma \perp L^2(G)_\pi$ if $\sigma, \pi \in \Pi$ are not G-isomorphic. The decomposition (3.22) follows from the denseness of matrix coefficients in $L^2(G)$. $\square$

An important consequence of Theorem 3.5 is the uniqueness of the Peter-Weyl decomposition for the regular representation.

Corollary 3.3. *Suppose that*

$$L^2(G) = \bigoplus_{i\in\mathcal{I}} W_i \tag{3.23}$$

is an orthogonal decomposition of $L^2(G)$ into T-irreducible subspaces. Then each component W_i occurs for precisely $d_i \triangleq \dim W_i$ times (up to G-isomorphism).

Proof. Exactly the same argumet for Theorem 3.1, Part (ii) and Part (iii) shows that all unitary irreducible representations are contained in the decomposition (3.23) and occur for at most finitely many times. As a result, one can rewrite the given decomposition as

$$L^2(G) = \bigoplus_{[(\pi,H_\pi)]\in\Pi} \big(W_1^\pi \oplus \cdots \oplus W_{r_\pi}^\pi\big),$$

where $W_1^\pi, \cdots, W_{r_\pi}^\pi$ are G-isomorphic to H_π for each $[(\pi, H_\pi)] \in \Pi$. It is sufficient to show that

$$W_1^\pi \oplus \cdots \oplus W_{r_\pi}^\pi = L^2(G)_\pi. \tag{3.24}$$

To this end, the key observation is that $W_i^\pi \subseteq L^2(G)_\pi$ $(1 \leqslant i \leqslant r_\pi)$. Indeed, let $\Phi_i : H_\pi \to W_i^\pi$ be a G-isomorphism. We regard Φ_i as a linear embedding into $L^2(G)$. Then for any $v \in H_\pi$ and $\varphi \in \mathcal{C}(G)$, one has

$$\begin{aligned}
\langle \pi(x)\Phi_i^*\varphi, v\rangle_{H_\pi} &= \langle \varphi, \Phi_i\pi(x^{-1})v\rangle_{L^2} \\
&= \int_G \varphi(y)\overline{\Phi_i(\pi(x^{-1})v)(y)}dy \\
&= \int_G \varphi(y)\overline{T(x^{-1})(\Phi_i v)(y)}dy \\
&= \int_G \varphi(y)\overline{(\Phi_i v)(yx^{-1})}dy \\
&= \big((\Phi_i v)^* * \varphi\big)(x),
\end{aligned} \tag{3.25}$$

where we recall that $\psi^*(x) \triangleq \overline{\psi(x^{-1})}$. Since the left hand side of (3.25) is an element of $L^2(G)_\pi$ (for every $\varphi \in \mathcal{C}(G)$), by approximation of identity one sees that $(\Phi_i v)^* \in L^2(G)_\pi$. But one also knows that $L^2(G)_\pi$ is $(\cdot)^*$-closed:

$$f(x) = \langle \pi(x)v, w\rangle_{H_\pi} \in L^2(G)_\pi \implies f^*(x) = \langle \pi(x)w, v\rangle_{H_\pi} \in L^2(G)_\pi.$$

Therefore, $\Phi_i v \in L^2(G)_\pi$. This shows that $W_i^\pi = \Phi_i(H_\pi) \subseteq L^2(G)_\pi$. In particular,

$$W_1^\pi \oplus \cdots \oplus W_{r_\pi}^\pi \subseteq L^2(G)_\pi,$$

and one also has $r_\pi \leqslant d_\pi$ since $\dim L^2(G)_\pi = d_\pi^2$. Theorem 3.5 (ii) implies $r_\pi = d_\pi$ and thus (3.24) holds. □

As a result of Corollary 3.3, $L^2(G)$ is isometrically isomorphic to the orthogonal direct sum of all unitary irreducible representations of G, each occurring with multiplicity equal to its dimension. Formally, one can write

$$L^2(G) = \bigoplus_{[(\pi, H_\pi)] \in \Pi} d_\pi H_\pi.$$

3.4.2 The Fourier inversion formula and Plancherel's theorem

We now make precise the Fourier inversion formula (3.2) and Plancherel's theorem (3.3) stated in the introduction of this section.

Let (π, H_π) be a unitary irreducible representation of G. The space $\mathcal{L}(H_\pi)$ of bounded linear operators on H_π is a d_π^2-dimensional Hilbert space under the Hilbert-Schmidt inner product

$$\langle S, T\rangle_{\mathrm{HS}} \triangleq \mathrm{Tr}(ST^*).$$

Define a linear map $\iota_\pi : \mathcal{L}(H_\pi) \to L^2(G)$ by

$$\iota_\pi(S) \triangleq [x \mapsto f_S(x) \triangleq \langle S, \pi(x)^*\rangle_{\mathrm{HS}}].$$

By definition,

$$f_S(x) = \mathrm{Tr}(S\pi(x)) = \mathrm{Tr}(\pi(x)S) = \sum_{i=1}^{d_\pi} \langle \pi(x)Se_i, e_i\rangle_{H_\pi}$$

where $\{e_1, \cdots, e_{d_\pi}\}$ is an ONB of H_π. In particular, $f_S \in L^2(G)_\pi$.

Lemma 3.8. *The linear map $d_\pi^{1/2}\iota_\pi : \mathcal{L}(H_\pi) \to L^2(G)_\pi$ is an isometric isomorphism.*

Proof. Since $\dim H_\pi = \dim L^2(G)_\pi = d_\pi^2$, one only needs to check that

$$\langle \iota_\pi S, \iota_\pi T\rangle_{L^2} = \frac{1}{d_\pi}\langle S, T\rangle_{\mathrm{HS}}, \quad S, T \in \mathcal{L}(H_\pi). \tag{3.26}$$

As both sides of (3.26) are bilinear in (S, T), it is enough to verify the identity for S, T having the form

$$S(v) = \lambda(v)e_i,\ T(v) = \mu(v)e_j,$$

where $\lambda, \mu \in H_\pi^*$ and $\{e_1, \cdots, e_{d_\pi}\}$ is an ONB of H_π.

By definition, one has

$$(\iota_\pi S)(x) = \mathrm{Tr}(\pi(x)S) = \sum_k \langle \pi(x)S(e_k), e_k\rangle_{H_\pi} = \sum_k \lambda(e_k)\langle \pi(x)e_i, e_k\rangle.$$

Therefore,

$$\begin{aligned}
\langle \iota_\pi S, \iota_\pi T\rangle_{L^2} &= \sum_{k,l} \lambda(e_k)\overline{\mu(e_l)} \int_G \langle \pi(x)e_i, e_k\rangle_{H_\pi} \overline{\langle \pi(x)e_j, e_l\rangle_{H_\pi}} dx \\
&= \sum_{k,l} \lambda(e_k)\overline{\mu(e_l)}\langle \pi_{ik}, \pi_{jl}\rangle_{L^2} \\
&= \frac{1}{d_\pi}\sum_k \lambda(e_k)\overline{\mu(e_k)}\delta_{ij},
\end{aligned}$$

where the last equality follows from the relation (3.17). On the other hand,

$$\begin{aligned}\langle S,T\rangle_{\mathrm{HS}} &= \mathrm{Tr}(ST^*) = \mathrm{Tr}(T^*S)\\ &= \sum_k \langle T^*Se_k, e_k\rangle_{H_\pi} = \sum_k \langle Se_k, Te_k\rangle_{H_\pi}\\ &= \sum_k \lambda(e_k)\overline{\mu(e_k)}\langle e_i, e_j\rangle_{H_\pi} = \sum_k \lambda(e_k)\overline{\mu(e_k)}\delta_{ij}.\end{aligned}$$

The identity (3.26) thus follows. □

We regard ι_π as a linear embedding from $\mathcal{L}(H_\pi)$ into $L^2(G)$. Using ι_π and its adjoint $\iota_\pi^* : L^2(G) \to \mathcal{L}(H_\pi)$, one can write down the orthogonal projection $P_\pi : L^2(G) \to L^2(G)_\pi$ precisely.

Proposition 3.10. *The projection P_π is given by*

$$P_\pi f = d_\pi \iota_\pi \iota_\pi^* f = d_\pi f * \chi_\pi, \tag{3.27}$$

where the operator ι_π^* is explicitly given by

$$\iota_\pi^* f = \pi^1(f^-) = \int_G f(x^{-1})\pi(x)dx \tag{3.28}$$

and χ_π is the character of π.

Proof. The first part of equation (3.27) is an immediate consequence of Lemma 3.8. In addition, for each $S \in \mathcal{L}(H_\pi)$ and $f \in L^2(G)$, one has

$$\begin{aligned}\langle f, \iota_\pi S\rangle_{L^2} &= \int_G f(x)\overline{(\iota_\pi S)(x)}dx = \int_G f(x)\overline{\mathrm{Tr}(S\pi(x))}dx\\ &= \int_G f(x)\mathrm{Tr}((S\pi(x))^*)dx = \int_G f(x)\mathrm{Tr}(\pi(x)^*S^*)dx\\ &= \langle \int_G f(x)\pi(x)^*dx, S\rangle_{\mathrm{HS}} = \langle \int_G f(x^{-1})\pi(x)dx, S\rangle_{\mathrm{HS}}.\end{aligned}$$

Therefore, the equation (3.28) holds. The second part of equation (3.27) follows from the fact that

$$\begin{aligned}(\iota_\pi \iota_\pi^* f)(x) &= \mathrm{Tr}(\iota_\pi^* f \cdot \pi(x)) = \int_G f(y)\mathrm{Tr}(\pi(y)^*\pi(x))dy\\ &= \int_G f(y)\mathrm{Tr}(\pi(y^{-1}x))dy = \int_G f(y)\chi_\pi(y^{-1}x)dy\\ &= (f * \chi_\pi)(x).\end{aligned}$$

□

Remark 3.5. From Schur's orthogonality relation (3.19), one has

$$\iota_\pi^* f = \pi^1(f^-) = \begin{cases} 0, & f \in L^2(G)_\pi^\perp; \\ \frac{\lambda(\cdot)v}{d_\pi}, & f = \pi_{\lambda,v} \in L^2(G)_\pi. \end{cases}$$

Definition 3.11. Let $f \in L^2(G)$. For each unitary irreducible representation (π, H_π) of G, the operator $\iota_\pi^* f \in \mathcal{L}(H_\pi)$ is called the *Fourier coefficient* of f at π and is denoted as $\hat{f}(\pi)$.

Next, we establish the Fourier inversion formula and Plancherel's theorem. Let $f \in L^2(G)$ and let $\{\hat{f}(\pi) : [\pi] \in \Pi\}$ be its collection of Fourier coefficients. Note that we pick one representative for each isomorphism class $[\pi] \in \Pi$.

Theorem 3.6. *(i) The Fourier inversion formula:*

$$f(x) = \sum_{[\pi]\in\Pi} d_\pi \langle \hat{f}(\pi), \pi(x)^* \rangle_{\mathrm{HS}} \quad \text{in } L^2(G).$$

(ii) Plancherel's theorem:

$$\|f\|_{L^2}^2 = \sum_{[\pi]\in\Pi} d_\pi \|\hat{f}(\pi)\|_{\mathrm{HS}}^2. \tag{3.29}$$

Proof. The inversion formula follows immediately from Theorem 3.5 and Proposition 3.10. To prove Plancherel's theorem, note from Lemma 3.8 that $d_\pi^{1/2}\iota_\pi : \mathcal{L}(H_\pi) \to L^2(G)_\pi$ is an isometry with inverse $d_\pi^{1/2}\iota_\pi^*$. As a result, one has

$$\|P_\pi f\|_{L^2}^2 = \|(d_\pi^{1/2}\iota_\pi^*)P_\pi f\|_{\mathrm{HS}}^2 = d_\pi \|\hat{f}(\pi)\|_{\mathrm{HS}}^2.$$

The identity (3.29) thus follows. □

Remark 3.6. Using the second part of the equation (3.27), one can express the inversion formula as

$$f(x) = \sum_{[\pi]\in\Pi} d_\pi (f * \chi_\pi)(x).$$

In addition, from the relation (3.28) one can also write Plancherel's theorem as

$$\|f\|_{L^2}^2 = \sum_{[\pi]\in} d_\pi \mathrm{Tr}(\pi^1(f^-)\pi^1(f^-)^*).$$

To conclude the theory, we examine a special class of functions in $L^2(G)$, for which the $L^2(G)$-expansion takes a particularly nice form in terms of characters.

Definition 3.12. A function $f \in L^2(G)$ is said to be a *class function*, if

$$f(yxy^{-1}) = f(x)$$

for all $x, y \in G$. The closed subspace of class functions in $L^2(G)$ is denoted as $L^2(G)^G$.

It is clear that characters are class functions. In addition, one has the following characterisation of class functions.

Lemma 3.9. *A function $f \in L^2(G)$ is a class function if and only if its Fourier coefficient $\hat{f}(\pi)$ is a G-homomorphism for each $[\pi] \in \Pi$.*

Proof. Suppose that f is a class function. Then

$$\begin{aligned} \pi(y)\hat{f}(\pi)\pi(y^{-1}) &= \int_G f(x)\pi(y)\pi(x)^*\pi(y^{-1})dx \\ &= \int_G f(x)\pi(yx^{-1}y^{-1})dx \\ &= \int_G f(yzy^{-1})\pi(z^{-1})dz = \hat{f}(\pi). \end{aligned}$$

Therefore, $\hat{f}(\pi)$ commutes with G-actions. Conversely, by using the inversion formula, one has

$$\begin{aligned} f(yxy^{-1}) &= \sum_{[\pi]\in\Pi} d_\pi \langle \hat{f}(\pi), \pi(yxy^{-1})^* \rangle_{\mathrm{HS}} \\ &= \sum_{[\pi]\in\Pi} d_\pi \mathrm{Tr}\big(\hat{f}(\pi)\pi(y)\pi(x)\pi(y^{-1})\big) \\ &= \sum_{[\pi]\in\Pi} d_\pi \mathrm{Tr}\big(\pi(y^{-1})\hat{f}(\pi)\pi(y)\pi(x)\big) \\ &= \sum_{[\pi]\in\Pi} d_\pi \mathrm{Tr}(\hat{f}(\pi)\pi(x)) = f(x). \end{aligned}$$

□

Let $f \in L^2(G)^G$ be a class function. Since $\hat{f}(\pi)$ is a G-homomorphism, from Proposition 3.2, Part (ii), one knows that $\hat{f}(\pi) = c_\pi \cdot \mathrm{Id}$ for some $c_\pi \in \mathbb{C}$. The constant c_π can be found by taking trace on both sides:

$$c_\pi \cdot d_\pi = \mathrm{Tr}(\hat{f}(\pi)) = \int_G f(x)\mathrm{Tr}(\pi(x)^*)dx = \langle f, \chi_\pi \rangle_{L^2}.$$

Therefore, $c_\pi = \frac{1}{d_\pi}\langle f, \chi_\pi\rangle_{L^2}$. The inversion formula for f becomes

$$f = \sum_{[\pi]\in\Pi} d_\pi \langle c_\pi \mathrm{Id}, \pi(x)^* \rangle_{\mathrm{HS}} = \sum_{[\pi]\in\Pi} \langle f, \chi_\pi\rangle_{L^2}\chi_\pi.$$

As a consequence, the family $\{\chi_\pi : [\pi] \in \Pi\}$ form an ONB of $L^2(G)^G$.

3.5 Two examples: the torus and the special unitary group SU(2)

We use two basic examples to illustrate the previous general theory. The first one is the abelian case in which the theory reduces to classical Fourier series. The second one is the simplest non-abelian example in which calculations can be performed explicitly.

3.5.1 The torus $\mathbb{T}^d$: classical Fourier series

The *d-dimensional torus* is the compact Lie group

$$\mathbb{T}^d \triangleq \underbrace{S^1 \times \cdots \times S^1}_{d \text{ times}} = \{(e^{i\theta_1}, \cdots, e^{i\theta_d}) : (\theta_1, \cdots, \theta_d) \in [0, 2\pi)^d\}.$$

Note that

$$\mathbb{T}^d \cong (\mathbb{R}/\mathbb{Z})^d \cong \mathbb{R}^d/\mathbb{Z}^d.$$

The torus $\mathbb{T}^d$ is an abelian group. Let $d\theta$ denote the normalised Haar measure on S^1. If $d_+\theta$ denotes the Lebesgue measure on $[0, 2\pi)$, then $d\theta = \frac{1}{2\pi} d_+\theta$. The normalised Haar measure on $\mathbb{T}^d$ is given by $d\theta_1 \times \cdots \times d\theta_d$.

The following result is quoted from the classical theory of Fourier series. We parametrise $\mathbb{T}^d$ by $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_d) \in [0, 2\pi)^d$.

Theorem 3.7. *Every $f \in L^2(\mathbb{T}^d)$ admits the following L^2-expansion:*

$$f(\boldsymbol{\theta}) = \sum_{\boldsymbol{k}=(k_1,\cdots,k_d)\in\mathbb{Z}^d} c^f_{\boldsymbol{k}} \cdot e^{i\langle \boldsymbol{k}, \boldsymbol{\theta}\rangle},$$

where

$$c^f_{\boldsymbol{k}} \triangleq \langle f, e^{i\langle \boldsymbol{k}, \cdot\rangle}\rangle_{L^2} = \int_{T^d} f(\boldsymbol{\theta}) e^{-i\langle \boldsymbol{k}, \boldsymbol{\theta}\rangle} d\boldsymbol{\theta}. \tag{3.30}$$

In addition, the following identity holds:

$$\|f\|^2_{L^2} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left|c^f_{\boldsymbol{k}}\right|^2. \tag{3.31}$$

To see how this classical result fits into the previous general theory, first note that every $\boldsymbol{k} \in \mathbb{Z}^d$ gives rise to a (one dimensional) unitary irreducible representation

$$\pi_{\boldsymbol{k}} : \mathbb{T}^d \to S^1,\ \pi_{\boldsymbol{k}}(\boldsymbol{\theta}) = e^{i\langle \boldsymbol{k}, \boldsymbol{\theta}\rangle}, \tag{3.32}$$

where the action on $\mathbb{C}$ is given by the complex multiplication. Indeed, these are all the possible unitary irreducible representations of $\mathbb{T}^d$.

Proposition 3.11. *Let $\pi : \mathbb{T}^d \to \mathrm{Aut}(H)$ be a unitary irreducible representation of $\mathbb{T}^d$. Then H is one dimensional and π is G-isomorphic to one of the representations $\pi_{\boldsymbol{k}}$ given by (3.32).*

Proof. Recall that H is finite dimensional. Since $\mathbb{T}^d$ is abelian and $\pi(x)$ is unitary on H, from linear algebra one knows that the family $\mathcal{A} \triangleq \{\pi(x) : x \in \mathbb{T}^d\}$ is simultaneously diagonalisable. In particular, H decomposes into an orthogonal direct sum of one dimensional common eigenspaces for $\mathcal{A}$. Since π is irreducible, one concludes that H has to be one dimensional. To reach the second assertion, by unitarity for each $\boldsymbol{\theta} \in \mathbb{T}^d$ there is a $\lambda(\boldsymbol{\theta}) \in \mathbb{R}$ (mod $2\pi\mathbb{Z}$) such that

$$\pi(\boldsymbol{\theta})v = e^{i\lambda(\boldsymbol{\theta})}v, \quad \forall \boldsymbol{\theta} \in \mathbb{T}^d \text{ and } v \in H.$$

This function λ satisfies

$$\lambda(\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2) = \lambda(\boldsymbol{\theta}_1) + \lambda(\boldsymbol{\theta}_2) \mod 2\pi\mathbb{Z},$$

which implies $\lambda(\boldsymbol{\theta}) = \langle \boldsymbol{k}, \boldsymbol{\theta}\rangle$ (mod $2\pi\mathbb{Z}$) for some $\boldsymbol{k} \in \mathbb{Z}^d$ (why?). Therefore, $\pi = \pi_{\boldsymbol{k}}$. □

We can now interpret Theorem 3.7 in the context of the general Theorem 3.6. The class of unitary irreducible representations are indexed by $\boldsymbol{k} \in \mathbb{Z}^d$. The Fourier coefficient of f at each $\boldsymbol{k}$ is given by

$$\hat{f}(\boldsymbol{k}) = \int_{\mathbb{T}^d} f(\boldsymbol{\theta})\pi_{\boldsymbol{k}}(\boldsymbol{\theta})^* d\boldsymbol{\theta} = \int_{\mathbb{T}^d} f(\boldsymbol{\theta})e^{-i\langle \boldsymbol{k},\boldsymbol{\theta}\rangle} d\boldsymbol{\theta} = c^f_{\boldsymbol{k}},$$

which is equivalently viewed as an operator on $\mathbb{C}$ acting by complex multiplication. In addition, one has $d_{\pi_{\boldsymbol{k}}} = 1$ and

$$\langle \hat{f}(\boldsymbol{k}), \pi_{\boldsymbol{k}}(\boldsymbol{\theta})^*\rangle_{\mathrm{HS}} = c^f_{\boldsymbol{k}} \cdot \mathrm{Tr}(\pi_{\boldsymbol{k}}(\boldsymbol{\theta})) = c^f_{\boldsymbol{k}} e^{i\langle \boldsymbol{k},\boldsymbol{\theta}\rangle}.$$

Therefore, the identities (3.30) and (3.31) are precisely the Fourier inversion formula and Plancherel's theorem. The function $e^{i\langle \boldsymbol{k},\boldsymbol{\theta}\rangle}$ is also the character of $\pi_{\boldsymbol{k}}$ and all functions in $L^2(\mathbb{T}^d)$ are class functions due to commutativity.

Remark 3.7. It is known that any compact, connected, abelian Lie group with dimension d is isomorphic to the torus $\mathbb{T}^d$.

3.5.2 SU(2): non-abelian Fourier analysis

We now consider the simplest example of a non-abelian compact Lie group on which the Fourier analysis can be worked out explicitly.

Definition 3.13. The *special unitary group of degree two* is the group of 2×2 unitary matrices with determinant one:

$$\mathrm{SU}(2) \triangleq \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The geometry of SU(2) and its Haar measure

From its definition, SU(2) is canonically diffeomorphic to the 3-sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

The identification is given by

$$\Phi : S^3 \ni (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \in \mathrm{SU}(2).$$

Lemma 3.10. *Let $g \in \mathrm{SU}(2)$. Define its action on S^3 from the right by*

$$R_g(x) \triangleq \Phi^{-1}(\Phi(x)g), \quad x \in S^3.$$

Then R_g is a special orthogonal transformation, i.e. $R_g \in \mathrm{SO}(4)$.

Proof. We extend the map Φ to the entire $\mathbb{R}^4$ which becomes a linear isomorphism onto its image. It is obvious that

$$\det \Phi(x) = \|x\|_{\mathbb{R}^4}^2 \quad \forall x \in \mathbb{R}^4.$$

Therefore,

$$\|R_g(x)\|_{\mathbb{R}^4}^2 = \det(\Phi(x)g) = \det(\Phi(x)) \det(g) = \|x\|_{\mathbb{R}^4}^2.$$

This shows that $R_g \in \mathrm{O}(4)$. Since $\mathrm{SU}(2) \cong S^3$ is connected, $g \mapsto R_g$ is continuous and $R_e = \mathrm{Id}$, one sees that $R_g \in \mathrm{SO}(4)$. $\square$

To write down the Haar measure explicitly, we introduce the following parametrisation (under the identification Φ):

$$\begin{cases} x_1 = \cos\theta, & x_3 = \sin\theta\sin\varphi\cos\psi, \\ x_2 = \sin\theta\cos\varphi, & x_4 = \sin\theta\sin\varphi\sin\psi, \end{cases} \tag{3.33}$$

where $0 < \theta < \pi$, $0 < \varphi < \pi$ and $0 < \psi < 2\pi$. This is obtained by setting $x_1 \triangleq \cos\theta$ and thinking of (x_2, x_3, x_4) as a generic point on the 2-sphere with radius $\sin\theta$. The parametrisation of (x_2, x_3, x_4) is the standard spherical parametrisation. Note that (θ, φ, ψ) does not provide a global coordinate system for $\mathrm{SU}(2)$. Nonetheless, the points that are missed out form a low dimensional manifold which does not affect calculations related to Haar measures and Haar integrals.

Under the coordinates (3.33), the standard Riemannian metric on S^3 is found to be

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= d\theta^2 + \sin^2\theta d\varphi^2 + \sin^2\theta\sin^2\varphi d\psi^2. \end{aligned}$$

As a result, the volume form on S^3 is given by

$$\mu = \sqrt{\det(g_{ij})}d\theta d\varphi d\psi = \sin^2\theta\sin\varphi d\theta d\varphi d\psi.$$

According to Lemma 3.10, $\mathrm{SU}(2)$ acts on S^3 (from the right) as isometries, hence leaving μ invariant. Therefore, when viewed as a measure on $\mathrm{SU}(2)$ (under the identification Φ) it is right invariant. After normalisation, one obtains the following fact.

Proposition 3.12. *Under the coordinates (3.33), the normalised Haar measure on* $\mathrm{SU}(2)$ *is given by*

$$dx = \frac{1}{2\pi^2}\sin^2\theta\sin\varphi d\theta d\varphi d\psi.$$

Classification of unitary irreducible representations

Our next goal is to classify all unitary irreducible representations of $\mathrm{SU}(2)$. We first write down the natural ones and then show that these are the only possibilities.

For each non-negative integer n, let H_n denote the space of degree n homogeneous polynomials in two complex variables. More explicitly,

$$H_n = \Big\{ \sum_{k=0}^{n} a_k z_1^k z_2^{n-k} : (a_0, \cdots, a_n) \in \mathbb{C}^{n+1} \Big\}.$$

It is clear that $\dim H_n = n+1$. A natural action of $\mathrm{SU}(2)$ on H_n is defined by

$$\pi_n(g)(\varphi)(z) \triangleq \varphi(zg), \quad g \in \mathrm{SU}(2), \varphi \in H_n.$$

Here $z = (z_1, z_2)$ is a row vector and zg is the usual matrix multiplication. A simple unwinding of definition shows that π_n is a representation of $\mathrm{SU}(2)$ on H_n. We define an inner product on H_n by

$$\langle (a_0, \cdots, a_n), (b_0, \cdots, b_n) \rangle_{H_n} \triangleq \sum_{k=0}^{n} k!(n-k)! a_k \overline{b_k}. \tag{3.34}$$

The reason for introducing the coefficients $k!(n-k)!$ is to make π_n into a unitary representation.

Lemma 3.11. *Under the inner product structure (3.34), $\pi_n : \mathrm{SU}(2) \to \mathrm{Aut}(H_n)$ is a unitary representation.*

Proof. For each column vector $a = (a_1, a_2)^T \in \mathbb{C}^2$, we define $\varphi_a \in H_n$ by

$$\varphi_a(z) \triangleq (za)^n = (a_1 z_1 + a_2 z_2)^n.$$

Then

$$\begin{aligned} \langle \varphi_a, \varphi_b \rangle_{H_n} &= \langle (a_1 z_1 + a_2 z_2)^n, (b_1 z_1 + b_2 z_2)^n \rangle_{H_n} \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 k!(n-k)! a_1^k a_2^{n-k} \overline{b_1}^k \overline{b_2}^{n-k} \\ &= n!(a_1 \overline{b_1} + a_2 \overline{b_2})^n = n! \langle a, b \rangle_{\mathbb{C}^2}^n. \end{aligned}$$

In addition, by definition one has

$$(\pi_n(g)\varphi_a)(z) = \varphi_a(zg) = (zga)^n = \varphi_{ga}(z).$$

Therefore,

$$\begin{aligned} &\langle \pi_n(g)\varphi_a, \pi_n(g)\varphi_b \rangle_{H_n} \\ &= \langle \varphi_{ga}, \varphi_{gb} \rangle_{H_n} = n! \langle ga, gb \rangle_{\mathbb{C}^2}^n \\ &= n! \langle a, b \rangle_{\mathbb{C}^2}^n = \langle \varphi_a, \varphi_b \rangle_{H_n}. \end{aligned}$$

This implies that π_n is unitary on the subspace generated by the φ_a's.

It remains to show that, the family $\{\varphi_a : a \in \mathbb{C}^2\}$ contains a basis of H_n so that it generates H_n. In fact, we claim that the $n+1$ functions

$$(z_1 + \omega^k z_2)^n \ (k = 0, 1, \cdots, n-1) \text{ and } z_2^n$$

form a basis of H_n, where $\omega \triangleq e^{2\pi i/n}$. One only needs to show their linear independence. Suppose that

$$\sum_{k=0}^{n-1} c_k (z_1 + \omega^k z_2)^n + c_n z_2^n = 0$$

with some constants $c_0, \cdots, c_n$. By considering the coefficient of $z_1^j z_2^{n-j}$ for each $0 \leqslant j \leqslant n$, one obtains the following linear system:

$$\begin{cases} \sum_{k=0}^{n-1} c_k \omega^{jk} = 0, & j = 0, 1, \cdots, n-1; \\ \sum_{k=0}^{n} c_k = 0. \end{cases}$$

After expanding along the last column, the determinant of the coefficient matrix is precisely the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{vmatrix} = \prod_{0 \leqslant j < k \leqslant n-1} (\omega^k - \omega^j) \neq 0.$$

Therefore, $c_0 = c_1 = \cdots = c_k = 0$. $\square$

By using the criterion given by Lemma 3.2, one can show that π_n is irreducible.

Lemma 3.12. *The representation π_n is irreducible.*

Proof. According to Lemma 3.2, one only needs to show that every SU(2)-homomorphism on H_n is a scalar multiple of the identity map. Let A be such a homomorphism. For each $0 \leqslant k \leqslant n$ we set $\varphi_k(z) \triangleq z_1^k z_2^{n-k}$. Let K be the subgroup of SU(2) consisting of the matrices $h_a \triangleq \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $(a \in S^1)$. The action of h_a on H_n is simple to describe:

$$\big(\pi_n(h_a)\varphi_k\big)(z) = \varphi_k(z h_a) = a^{2k-n} \varphi_k(z). \tag{3.35}$$

Using this observation, it can be seen that a function $\varphi \in H_n$ is proportional to φ_k if and only if $\pi_n(h_a)\varphi = a^{2k-n}\varphi$ for all $a \in S^1$.

On the other hand, by the assumption one has

$$\pi_n(h_a)A\varphi_k = A\pi_n(h_a)\varphi_k = a^{2k-n}A\varphi_k \quad \forall a \in S^1.$$

Therefore, $A\varphi_k = c_k\varphi_k$ with some $c_k \in \mathbb{C}$. To show that all the c_k's are identical, we use another subgroup R consisting of the matrices $r_t \triangleq \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ $(t \in [0, 2\pi))$. Direct calculation shows that

$$A\pi_n(r_t)\varphi_n = A(z_1 \cos t + z_2 \sin t)^n = \sum_{k=0}^{n} \binom{n}{k} \big(\cos^k t \sin^{n-k} t\big) c_k \varphi_k,$$

while

$$\pi_n(r_t)A\varphi_n = c_n \sum_{k=0}^{n} \binom{n}{k} \big(\cos^k t \sin^{n-k} t\big) \varphi_k.$$

According to the assumption, one has

$$A\pi_n(r_t)\varphi_n = \pi_n(r_t)A\varphi_n \quad \forall t \in [0, 2\pi). \tag{3.36}$$

By comparing the coefficients of φ_k on both sides, the relation (3.36) is possible only when $c_k = c_n$ for all k. Therefore, $A\varphi_k = c_n\varphi_k$ for all k, showing that $A = c_n \cdot \mathrm{Id}$. $\square$

We now compute the character of π_n. Before doing so, let us first observe that every element $g \in \mathrm{SU}(2)$ is unitarily conjugate to a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Since g is unitary with determinant one, the eigenvalues a, b must have the form $a = e^{i\theta}$, $b = e^{-i\theta}$ for some θ. As a result, any class function f on $\mathrm{SU}(2)$ is uniquely determined by its values on the subgroup

$$K = \left\{ h_\theta \triangleq \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [-\pi, \pi] \right\}.$$

In other words, f can be viewed as a function of h_θ or θ. Such a function must also be an even function in θ, since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Proposition 3.13. *The character of π_n is given by*

$$\chi_n(h_\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \theta \in [-\pi, \pi]. \tag{3.37}$$

Proof. According to the relation (3.35), $\pi_n(h_\theta)$ acts on the basis $\{\varphi_k : 0 \leqslant k \leqslant n\}$ diagonally. Therefore,

$$\chi_n(h_\theta) = \mathrm{Tr}\big(\pi_n(h_\theta)\big) = \sum_{k=0}^{n} e^{i(2k-n)\theta} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n+1)\theta}{\sin\theta}.$$

□

Remark 3.8. When $\theta = 0$ or π, the quotient in (3.37) is understood in the limiting sense as $\theta \to 0$ or π. More explicitly, one has

$$\chi_n(1) = n+1, \ \chi_n(-1) = (-1)^n(n+1).$$

Lemma 3.13. *Let $f \in L^2(\mathrm{SU}(2))$ be a class function. Then*

$$\int_G f(x)dx = \frac{2}{\pi}\int_0^\pi f(h_\theta)\sin^2\theta d\theta.$$

Proof. The main observation is that

$$f(x) = f(\theta, 0, 0) = f(h_\theta), \tag{3.38}$$

where the first θ denotes the θ-coordinate of x under the parametrisation (3.33) while the second h_θ is the element to which x is unitarily conjugate. Indeed, since $\det x = 1$, the characteristic equation of x is given by $t^2 - \mathrm{Tr}(x)t + 1 = 0$, which under the (θ, φ, ψ)-coordinates reduces to

$$t^2 - 2t\cos\theta + 1 = 0.$$

It follows that the eigenvalues of x are given by $e^{\pm i\theta}$. In particular, x is unitarily conjugate to h_θ. Therefore, the relation (3.38) holds. The result follows from Proposition 3.12 by integrating out the φ and ψ variables. □

We can now establish the classification theorem for unitary irreducible representations of SU(2).

Theorem 3.8. *The representations π_n ($n \geqslant 0$) are all distinct. In addition, any non-trivial unitary irreducible representation of* SU(2) *is G-isomorphic to one of the π_n's.*

Proof. The first assertion is trivial since the dimensions of the H_n's are all different. For the second assertion, let (π, H) be a non-trivial unitary irreducible representation of SU(2). Suppose on the contrary that π is distinct from all the $\pi_n' s$. Let $\chi(\theta)$ be the character of π and define

$$\xi(\theta) \triangleq \chi(\theta)\sin\theta, \quad \theta \in [-\pi, \pi].$$

We use

$$\langle f_1(\theta), f_2(\theta)\rangle_{L^2(S^1)} \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\theta)\overline{f_2(\theta)}d\theta$$

to distinguish from the L^2-inner product on SU(2) for class functions, which is given by

$$\langle f_1(\theta), f_2(\theta)\rangle_{L^2(\mathrm{SU}(2))} = \frac{2}{\pi}\int_0^{\pi} f_1(\theta)\overline{f_2(\theta)}\sin^2\theta d\theta.$$

Since $\xi(\theta)$ is an odd function, elementary calculation together with the formula (3.37) show that

$$\langle \xi, e^{\pm in\theta}\rangle_{L^2(S^1)} = \frac{\mp i}{2}\langle \chi, \chi_{n-1}\rangle_{L^2(\mathrm{SU}(2))}$$

for each $n \geqslant 1$. In addition, one has $\langle \xi, 1\rangle_{L^2(S^1)} = 0$. Therefore, from the classical Parseval's identity (3.31) one obtains

$$\|\xi\|_{L^2(S^1)}^2 = \sum_{n=-\infty}^{\infty}\left|\langle \xi, e^{in\theta}\rangle_{L^2(S^1)}\right|^2 = \frac{1}{2}\sum_{n=0}^{\infty}\left|\langle \chi, \chi_n\rangle_{L^2(\mathrm{SU}(2))}\right|^2.$$

On the other hand, one also has

$$\|\chi\|_{L^2(\mathrm{SU}(2))}^2 = \frac{2}{\pi}\int_0^{\pi}|\chi(\theta)|^2\sin^2\theta d\theta = 2\|\xi\|_{L^2(S^1)}^2.$$

As a result, one concludes that

$$\|\chi\|_{L^2(\mathrm{SU}(2))}^2 = \sum_{n=0}^{\infty}\left|\langle \chi, \chi_n\rangle_{L^2(\mathrm{SU}(2))}\right|^2.$$

Since π is distinct from π_n, from Schur's orthogonality relation (3.20) one has $\langle \chi, \chi_n\rangle_{L^2(\mathrm{SU}(2))} = 0$ for each n, hence yielding $\chi = 0$. This is a contradiction to the fact that $\langle \chi, \chi\rangle_{L^2(\mathrm{SU}(2))} = 1$ in view of the same relation (3.20). $\square$

Under the above classification, one can write down the corresponding Fourier identities. For instance, Plancherel's theorem reads

$$\|f\|_{L^2}^2 = \sum_{n=0}^{\infty}(n+1)\|\hat{f}(n)\|_{\mathrm{HS}}^2.$$

In addition, for any class function $f = f(\theta)$ one has

$$\langle f, \chi_n\rangle_{L^2(\mathrm{SU}(2))} = \frac{2}{\pi}\int_0^{\pi} f(t)\sin\big((n+1)t\big)\sin t dt.$$

Therefore, $\hat{f}(n) = c_n \mathrm{Id}$ where

$$c_n = \frac{1}{n+1}\langle f, \chi_n\rangle_{L^2(\mathrm{SU}(2))} = \frac{2}{(n+1)\pi}\int_0^{\pi} f(t)\sin\big((n+1)t\big)\sin t dt.$$

The inversion formula for f reads

$$f(\theta) = \sum_{n=0}^{\infty}\frac{2}{\pi}\Big(\int_0^{\pi} f(t)\sin(n+1)t\sin t dt\Big)\cdot\frac{\sin(n+1)\theta}{\sin\theta}$$

with convergence understood in the L^2-sense with respect to the measure $\frac{2}{\pi}\sin^2\theta d\theta$.

Remark 3.9. Under suitable smoothness conditions, it is possible to establish pointwise and uniform convergence results for the inversion formula. This requires deeper analysis for certain differential operators as well as Lie algebra considerations. In the SU(2)-case, we refer the reader to [5] for a discussion on this topic.

Remark 3.10. The representation theory for non-compact groups is much deeper than the compact case and one must also consider infinite dimensional representations apart from the finite dimensional ones. We refer the reader to [3] for a beautiful presentation in the simplest non-trivial example of $\mathrm{SL}_2(\mathbb{R})$, which is a real source of inspiration in many ways.

Appendix A Set theory: Zorn's lemma

In this appendix, we recall Zorn's lemma from set theory, which is used in the study of complete reducibility.

Let P be a given partially ordered set. A *chain* in P is a totally ordered subset of P. An *upper bound* of a subset $S \subseteq P$ is an element $y \in P$ such that $x \leqslant y$ for every $x \in S$. A *maximal element* in P is an element that is not smaller than any element in P.

Theorem A.1 (Zorn's lemma)**.** *If every chain in P has an upper bound in P, then P contains at least one maximal element.*

Appendix B General topology: Tychonoff's theorem and Urysohn's theorem

In this appendix, we collect two fundamental theorems in general topology that are used in the previous discussions of Radon measures and Haar integrals.

The first theorem concerns with product topology and compactness. Let $\{X_i : i \in \mathcal{I}\}$ be a family of topological spaces. Define the product space

$$X \triangleq \prod_{i \in \mathcal{I}} X_i.$$

The *product topology* on X is the coarsest topology (i.e. the topology with the fewest open sets) such that the canonical projections $\pi_i : X \to X_i$ $(i \in \mathcal{I})$ are continuous. This is the topology generated by the subsets of the form $p_i^{-1}(U_i)$ where U_i is open in X_i. The following theorem, known as *Tychonoff's theorem*, asserts that compactness is preserved under the product topology.

Theorem B.1. *Suppose that X_i is a compact space for every $i \in \mathcal{I}$. Then X is also compact under the product topology.*

The next theorem, known as *Urysohn's lemma*, allows us to construct sufficiently many continuous functions on topological spaces. Recall that a *normal space* is a topological space in which every two disjoint closed subsets have disjoint open neighbourhoods.

Theorem B.2. *Let F_1, F_2 be two disjoint closed subsets of a normal space X. Then there exists a continuous function $f : X \to [0,1]$ such that*

$$f = 0 \text{ on } F_1 \text{ and } f = 1 \text{ on } F_2.$$

Appendix C Functional analysis

In this appendix, we collect several basic theorems in functional analysis that are used in Section 3.

The first result is the *Uniform Boundedness Theorem* for bounded linear operators, which asserts that pointwise boundedness implies uniform boundedness.

Theorem C.1. *Let E be a Banach space, and let $\mathcal{A}$ be a family of bounded linear operators on E. Suppose that*

$$\sup_{A\in\mathcal{A}} \|A(v)\|_E < \infty$$

for each $v \in E$. Then

$$\sup_{A\in\mathcal{A}} \|A\|_{E\to E} < \infty.$$

The next result is known as the *Stone-Weierstrass theorem* (the complex version). It is rather useful when studying uniform approximations of continuous functions.

Theorem C.2. *Let X be a compact Hausdorff space. Let $\mathcal{A} \subseteq \mathcal{C}(X)$ be a family of complex-valued continuous functions satisfying the following properties:*

(i) $\mathcal{A}$ is an algebra which contains constant functions and is self-conjugate:

$$f, g \in \mathcal{A}, c \in \mathbb{C} \implies c, cf + g, fg, \overline{f} \in \mathcal{A}.$$

(ii) $\mathcal{A}$ separate points: for any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Then $\mathcal{A}$ is dense in $\mathcal{C}(X)$ under the uniform topology.

The third result is the spectral theorem for compact self-adjoint operators on Hilbert spaces. It generalises the classical spectral theorem for Hermitian matrices to the infinite dimensional situation.

Theorem C.3. *Let A be a compact self-adjoint operator on a Hilbert space H. Then the following statements hold true:*

(i) all eigenvalues of A are real, and for each $\eta > 0$ there are finitely many eigenvalues λ such that $|\lambda| \geqslant \eta$;

(ii) the eigenspace space E_λ corresponding to each non-zero eigenvalue λ is finite

dimensional;
(iii) the space H admits a decomposition into the orthogonal direct sum of eigenspaces:

$$H = \bigoplus_{\lambda} E_\lambda.$$

The last result we shall recall is the functional *Schur's lemma.* It is particularly useful in the study of infinite dimensional representations.

Theorem C.4. *Let $\mathcal{A}$ be a family of bounded linear operators on a Hilbert space H. Suppose that H is $\mathcal{A}$-irreducible. Let Q be a self-adjoint bounded linear operator that commutes with $\mathcal{A}$, i.e. $QA = AQ$ for all $A \in \mathcal{A}$. Then $Q = c \cdot \mathrm{Id}$ for some $c \in \mathbb{C}$.*

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